Math 142: Calculus II

Midterm 1

October 12, 2017

NAME (please print legibly): Solutions

Your University ID Number: ______________________

Indicate your instructor with a check in the appropriate box:

Crossen	MW 9-10:15	
Zhong	MW 3:25-4:40	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 9 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________

QUESTION	VALUE	SCORE
1	15	
2	10	
3	10	
4	15	
5	10	
6	15	
TOTAL	75	

1. (15 points) Consider the following function

$$f(x) = \frac{x^2+1}{x^2-9}, \quad f'(x) = -\frac{20x}{(x^2-9)^2}, \quad f''(x) = \frac{60x^2+180}{(x^2-9)^3}.$$

(a) Find the domain of f.

$x^2 - 9 \neq 0$

$(x-3)(x+3) \neq 0$

$x \neq \pm 3$

$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

(b) List all x and y intercepts of f.

y-intercept: $f(0) = \frac{0+1}{0-9} = -\frac{1}{9}$ $\boxed{(0, -1/9)}$

x-intercept: $\frac{x^2+1}{x^2-9} = 0$ no solutions

(c) Final all horizontal and vertical asymptotes of f or explain why none exist.

h.a.: $\lim_{x\to\infty} \frac{x^2+1}{x^2-9} = \lim_{x\to\infty} \frac{1 + 1/x^2}{1 - 9/x^2} = 1$ $\boxed{y = 1 \text{ h.a.}}$

$\lim_{x\to-\infty} \frac{x^2+1}{x^2-9} = \dots = 1$

v.a. candidates: $x = \pm 3$

$\lim_{x\to-3^-} \frac{x^2+1}{(x-3)(x+3)} = \frac{10}{(-6)(\text{small } -ve)} = +\infty$

$\lim_{x\to-3^+} \frac{x^2+1}{(x-3)(x+3)} = \frac{10}{(-6)(\text{small } +ve)} = -\infty$

$\lim_{x\to 3^-} \frac{x^2+1}{(x-3)(x+3)} = \frac{10}{(\text{small } -ve)(6)} = -\infty$

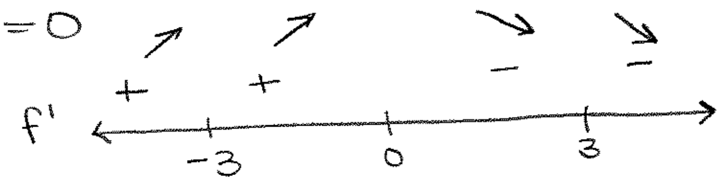
$\lim_{x\to 3^+} \frac{x^2+1}{(x-3)(x+3)} = \frac{10}{(\text{small } +ve)(6)} = \infty$

$\boxed{\text{v.a. } x = \pm 3}$

(d) Find all critical points and where the function is increasing and where it is decreasing.

$f'(x) = 0 = \dfrac{-20x}{(x^2-9)^2} \Rightarrow x = 0$

$f'(x)$ DNE at $x = \pm 3$
not critical points
(not in domain)

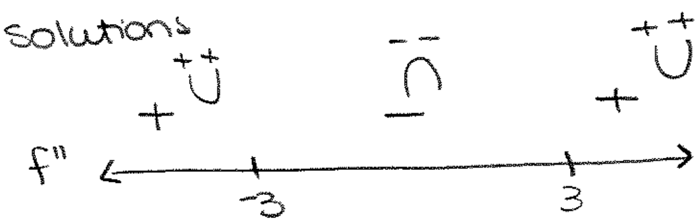


increasing: $(-\infty,-3) \cup (-3,0)$
decreasing: $(0,3) \cup (3,\infty)$
Critical points: $x = 0$

(e) Find where the function is concave up, and where it is concave down, and all x values at points of inflection.

$f''(x) = 0 = \dfrac{60x^2+180}{(x^2-9)^3}$ no solutions

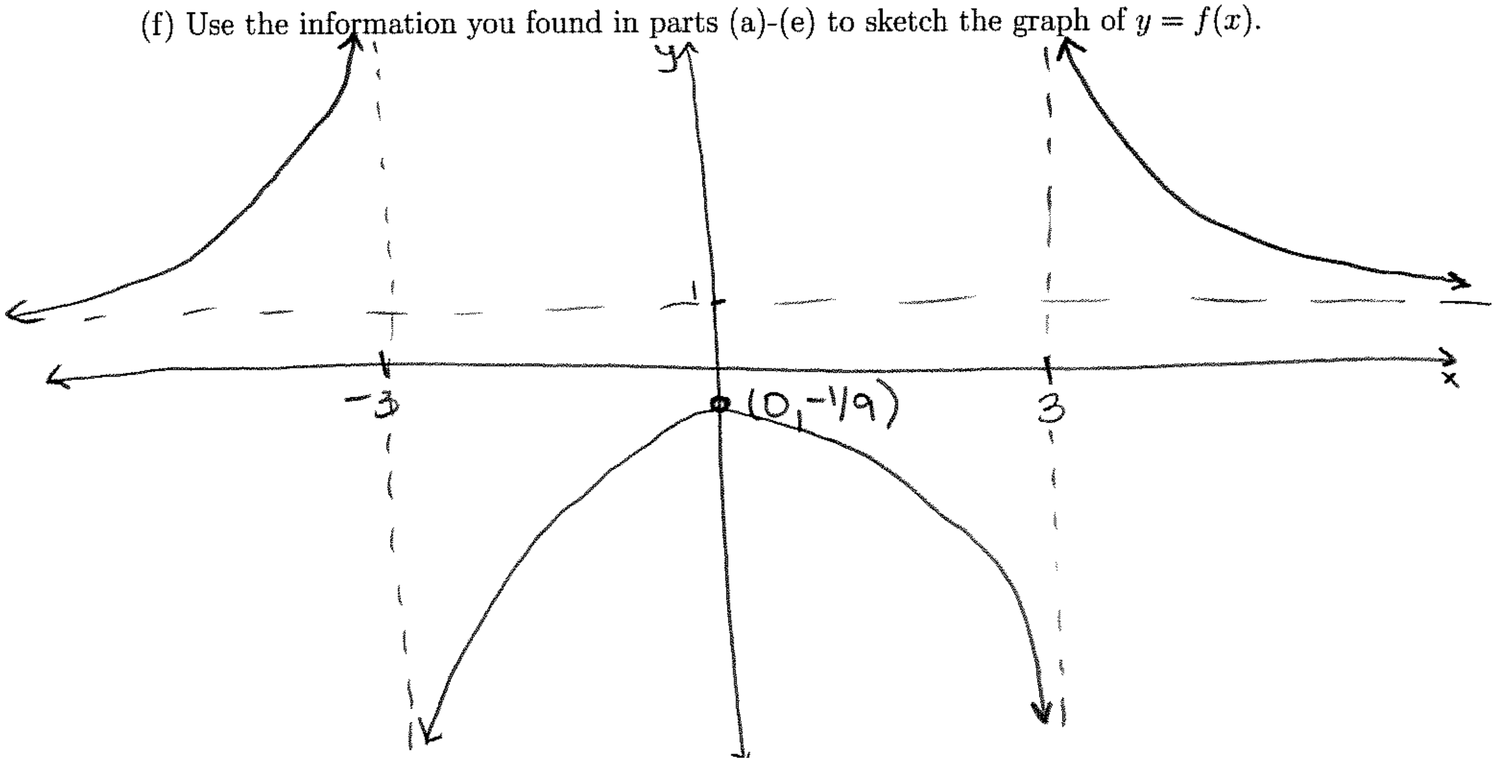
$f''(x)$ DNE at $x = \pm 3$
not inflection points,
not in domain.



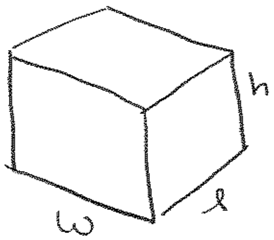
$f''(-100) = \frac{+}{+}$ $f''(0) = \frac{+}{-}$ $f''(100) = \frac{+}{+}$

no inflection points
up: $(-\infty,-3) \cup (3,\infty)$
down: $(-3,3)$

(f) Use the information you found in parts (a)-(e) to sketch the graph of $y = f(x)$.



2. (10 points) A manufacturer needs to make a box whose base length is twice the base width. The material used to build the top and bottom costs \$10/ft^2 and the material used to build the sides costs \$6/ft^2. If the box must have a volume of 960 ft^3, determine the dimensions that will minimize the cost to build the box.



$$C = 10(2w\ell) + 6(2\ell h + 2wh)$$

$$C = 20w(2w) + 12(2w)\left(\frac{480}{w^2}\right) + 12w\left(\frac{480}{w^2}\right)$$

$$C = 40w^2 + \frac{24\cdot 480}{w} + \frac{12\cdot 480}{w}$$

$$C = 40w^2 + 36\cdot 480 w^{-1}$$

$$\frac{dC}{dw} = 80w - 36\cdot 480 w^{-2} = 0$$

$$80w = \frac{36\cdot 480}{w^2}$$

$$w^3 = \frac{36\cdot 480}{80} = 36\cdot 6 = 6^3$$

$$w = 6$$

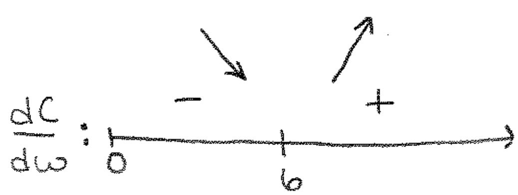
$\ell = 2w$

$\ell w h = 960$

$(2w)wh = 960$

$h = \frac{480}{w^2}$

<u>domain</u>: $0 < w < \infty$



$w = 6$ is absolute min

$$\ell = 2w = 12$$

$$h = \frac{480}{6^2} = \frac{80}{6} = \frac{40}{3}$$

$$\boxed{6 \times 12 \times \frac{40}{3} \text{ ft}}$$

3. (10 points) A particle is moving with the given acceleration

$$a(t) = 10\sin(t) + 3\cos(t),$$

and two positions

$$s(0) = 0, \quad s(2\pi) = 12.$$

Find the position function $s(t)$ of the particle.

$$v(t) = \int a(t)\,dt = \int (10\sin t + 3\cos t)\,dt$$

$$= -10\cos t + 3\sin t + C$$

$$s(t) = \int v(t)\,dt = \int (-10\cos t + 3\sin t + C)\,dt$$

$$s(t) = -10\sin t - 3\cos t + Ct + D$$

$$s(0) = 0 = 0 - 3(1) + 0 + D \Rightarrow D = 3$$

$$s(2\pi) = 12 = 0 - \cancel{3(1)} + 2\pi C + \cancel{3}$$

$$C = 6/\pi$$

$$\boxed{s(t) = -10\sin t - 3\cos t + \frac{6t}{\pi} + 3}$$

4. (15 points) Estimate the area between the graph of $y = (x+1)^2$ and x-axis from $x = 0$ to $x = 2$ using a right-endpoint Riemann sum with:

(a) 4 rectangles

$$R_4 = \Delta x\left(f(x_1) + f(x_2) + f(x_3) + f(x_4)\right)$$

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

$$= \frac{1}{2}\left[(1/2+1)^2 + (1+1)^2 + (3/2+1)^2 + (2+1)^2\right]$$

$$= \frac{1}{2}\left[(3/2)^2 + 2^2 + (5/2)^2 + 3^2\right]$$

$$= \frac{1}{2}\left[\frac{9}{4} + 4 + \frac{25}{4} + 9\right] = \frac{1}{2}\left[\frac{9}{4} + \frac{16}{4} + \frac{25}{4} + \frac{36}{4}\right]$$

$$= \frac{1}{2}\left[\frac{86}{4}\right] = \boxed{\frac{43}{4}}$$

(b) n rectangles (express your answer in terms of n)

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{2i}{n}+1\right)^2\frac{2}{n} = \sum_{i=1}^{n}\left(\frac{4i^2}{n^2} + \frac{4i}{n} + 1\right)\frac{2}{n}$$

$$\Delta x = \frac{2}{n}$$

$$x_i = \frac{2i}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} + \frac{8i}{n^2} + \frac{2}{n}\right) = \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2\right) + \frac{8}{n^2}\left(\sum_{i=1}^{n} i\right) + \frac{2}{n}\left(\sum_{i=1}^{n} 1\right)$$

$$= \frac{\overset{4}{\cancel{8}}}{n^{\cancel{3}2}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,3}\right) + \frac{\overset{4}{\cancel{8}}}{n^{\cancel{2}}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}}\right) + \frac{2}{n}(n)$$

$$= \boxed{\frac{4(n+1)(2n+1)}{3n^2} + \frac{4(n+1)}{n} + 2}$$

(c) Write the area as a define integral and use your answer in part (b) to determine the exact value of the area. **(No partial credit if you use other methods.)**

$$\int_0^2 (x+1)^2\,dx = \lim_{n\to\infty}\left[\frac{4(n+1)(2n+1)}{3n^2} + \frac{4(n+1)}{n} + 2\right]$$

$$= \lim_{n\to\infty}\left[\frac{8n^2+12n+4}{3n^2} + \frac{4n+1}{n} + 2\right]$$

$$= \frac{8}{3} + 4 + 2 = \boxed{26/3}$$

6. (15 points)

(a) Evaluate $\displaystyle\int_1^{64} \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[6]{x}}\,dx.$ $\displaystyle = \int_1^{64} \frac{x^{1/2}+x^{1/3}}{x^{1/6}}dx = \int_1^{64}(x^{1/3}+x^{1/6})dx$

$$=\left[\frac{3}{4}x^{4/3}+\frac{6}{7}x^{7/6}\right]_1^{64} = \left[\frac{3}{4}(64)^{4/3}+\frac{6}{7}(64)^{7/6}\right]-\left[\frac{3}{4}+\frac{6}{7}\right]$$

(b) Suppose that

$$f(x)=\int_{\arctan x}^{\sin(2\pi x)}\sqrt{1+t^2}\,dt \quad \text{and} \quad g(y)=\int_3^{2y} f(x)\,dx.$$

Find $g'(y)$, $g''(y)$ and $g''(0)$.

$$g'(y) = f(2y)\cdot 2 = \boxed{2\int_{\arctan(2y)}^{\sin(4\pi y)}\sqrt{1+t^2}\,dt = g'(y)}$$

$$g'(y) = 2\int_0^{\sin(4\pi y)}\sqrt{1+t^2}\,dt - 2\int_0^{\arctan(2y)}\sqrt{1+t^2}\,dt$$

$$g''(y) = 2\sqrt{1+\sin^2(4\pi y)}\,(\cos(4\pi y)\cdot 4\pi) - 2\sqrt{1+\arctan^2(2y)}\cdot\frac{1}{1+4y^2}(2)$$

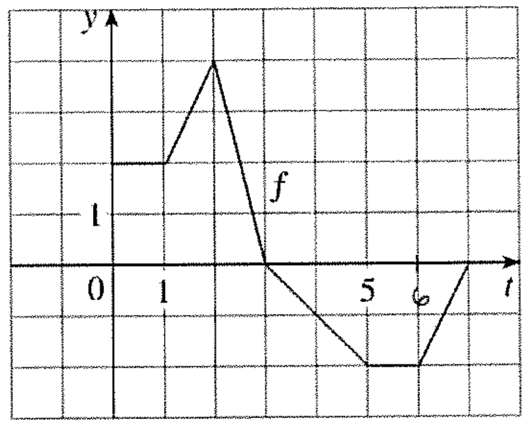
$$\boxed{g''(y) = 8\pi\cos(4\pi y)\sqrt{1+\sin^2(4\pi y)} - \frac{4\sqrt{1+\arctan^2(2y)}}{1+4y^2}}$$

$$g''(0) = 8\pi(1)(1) - \frac{4(1)}{1} = \boxed{8\pi - 4}$$

5. (10 points) Let

$$g(x) = \int_0^x f(t)\,dt, \quad x \in [0, 7].$$

where f is the function whose graph is shown.



(a) Evaluate $g(1)$, $g(2)$, and $g(6)$.

$g(1) = (1)(2) = \boxed{2}$

$g(2) = 2 + (1)(2) + \frac{1}{2}(1)(2) = \boxed{5}$

$g(6) = 5 + \frac{1}{2}(1)(4) - \frac{1}{2}(2)(2) - (1)(2)$

$= 7 - 2 - 2 = \boxed{3}$

(b) On what intervals is g increasing?

$[0, 3)$

(c) Where does g have a maximum value? What is the maximum value?

$t = 3$

$g(3) = 5 + \frac{1}{2}(1)(4) = 7$

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