MATH 142

Midterm 2 ANSWERS

April 1, 2014

1. (8 points)

Find

$$\frac{d}{dx}\int_1^{x^3} \cos t \; dt$$

Answer:

Let $K(x) = \int_1^x \cos t dt$, then $K'(x) = \cos x$ by FTC. Thus, by the chain rule

$$\frac{d}{dx}\int_1^{x^3} \cos t \; dt = \frac{d}{dx}K(x^3) = K'(x^3) \cdot 3x^2 = 3x^2 \cos x^3.$$

2. (48 points) Evaluate the following integrals.

(a) (8 points)

$$\int_0^1 \left(3\sqrt{x} - \frac{2}{1+x^2}\right) \; dx.$$

Answer:

(a)

$$\int_0^1 \left(3\sqrt{x} - \frac{2}{1+x^2}\right) dx = \left[2x^{3/2} - 2\arctan x\right]_0^1 = (2 - 2\arctan 1) - (0) = 2 - \frac{\pi}{2}.$$

(b) (8 points)

$$\int \frac{\sin\theta}{\cos^2\theta} \; d\theta.$$

Answer:

(b)

$$\int \frac{\sin\theta}{\cos^2\theta} d\theta = \int \tan\theta \sec\theta d\theta = \sec\theta + C.$$

(c) (8 points)

$$\int x^3(5-x^2)\ dx.$$

Answer:

(c)

$$\int x^3(5-x^2)dx = \int (5x^3 - x^5)dx = \frac{5}{4}x^4 - \frac{1}{6}x^6 + C.$$

(d) (8 points)

$$\int \frac{1}{x^2\sqrt{1+1/x}}\ dx.$$

Answer:

(d) Let $u = 1 + 1/x$, then $du = -(1/x^2)dx$. Hence

$$\int \frac{1}{x^2\sqrt{1+1/x}}dx = -\int \frac{1}{\sqrt{u}}du = -2\sqrt{u} + C = -2\sqrt{1+1/x} + C$$

(e) (8 points)

$$\int_0^2 2e^{x/2}\ dx.$$

Answer:

(e)

$$\int_0^2 2e^{x/2}dx = [4e^{x/2}]_0^2 = 4e - 4.$$

(f) (8 points)

$$\int_{-1}^1 |x^2 - x|\ dx.$$

Answer:

(f) Note that $x^2 - x = x(x-1) \leq 0$ if and only if $0 \leq x \leq 1$. Thus $|x^2 - x| = x^2 - x$ for

$-1 \leq x \leq 0$ and $|x^2 - x| = x - x^2$ for $0 \leq x \leq 1$. Therefore

$$\begin{aligned}\int_{-1}^{1} |x^2 - x| dx &= \int_{-1}^{0} (x^2 - x) dx + \int_{0}^{1} (x - x^2) dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{0}^{1} \\ &= -(-1/3 - 1/2) + (1/2 - 1/3) = 1.\end{aligned}$$

3. (16 points) An object is moving in such a way that its velocity function at time t is given by $v(t) = \sin(t)$.

(a) (8 points) Find the displacement from $t = 0$ to $t = 2\pi$.

Answer:

(a)

$$\int_{0}^{2\pi} \sin(t) dt = [-\cos(t)]_{0}^{2\pi} = -\cos(2\pi) - (-\cos(0)) = -1 - (-1) = 0$$

(b) (8 points) Find the total distance traveled from $t = 0$ to $t = 2\pi$.

Answer:

(b)

$$\begin{aligned}\int_{0}^{2\pi} |\sin(t)| dt &= \int_{0}^{\pi} \sin(t) dt - \int_{\pi}^{2\pi} \sin(t) dt \\ &= [-\cos(t)]_{0}^{\pi} - [-\cos(t)]_{\pi}^{2\pi} \\ &= (-\cos(\pi) - (-\cos(0))) - (-\cos(2\pi) - (-\cos(\pi))) \\ &= (1 + 1) - (-1 - 1) = 4\end{aligned}$$

4. (12 points)

Find the area of the region bounded by the curves $x = y^2$ and $x = 4y$.

Answer:

To find where the two curves intersect we solve the equations simultaneously. Set $y^2 = 4y$, which gives $y^2 - 4y = 0$ or $y(y - 4) = 0$ so $y = 0$ or $y = 4$. The points of intersection are thus $(0, 0)$ and $(16, 4)$. For $0 \leq y \leq 4$ we see that $4y$ is larger than y^2 and thus we get that the area between the two curves is

$$\int_0^4 (4y - y^2)dy = \left[2y^2 - \frac{1}{3}y^3\right]_0^4 = \left(2 \cdot 4^2 - \frac{1}{3} \cdot 4^3\right) - \left(2 \cdot 0^2 - \frac{1}{3} \cdot 0\right) = \frac{32}{3}$$

5. (16 points) Consider the region enclosed by the three curves $y = x^2$, $x = 2$ and $y = 0$.

(a) (8 points) Set up a definite integral that represents the volume of the solid obtained by rotating this region about $y = 7$. Do NOT evaluate the integral.

Answer:

(a) If you choose to use the method of washers you get

$$\int_0^2 \left(\pi \cdot 7^2 - \pi(7 - x^2)^2\right)\ dx.$$

If you choose to use the method of cylindrical shells you get

$$\int_0^4 2\pi(7 - y)(2 - \sqrt{y})\ dy.$$

(b) (8 points) Set up a definite integral that represents the volume of the solid obtained by rotating this region about $x = -1$. Do NOT evaluate the integral.

Answer:

(b) If you choose to use the method of cylindrical shells you get

$$\int_0^2 2\pi(x - (-1))\ x^2\ dx$$

If you choose to use the method of washers you get

$$\int_0^4 \left(\pi \cdot (2 - (-1))^2 - \pi(\sqrt{y} - (-1))^2\right)\ dy.$$