MATH 142

Midterm 1 ANSWERS

February 26, 2014

1. (24 points)

(a), (6 points) Find the vertical and horizontal asymptotes of

$$f(x) = \frac{2x^2 + x + 1}{x^2 - 2}.$$

Answer:

(a) The denominator factors into $(x - \sqrt{2})(x + \sqrt{2})$. Therefore the function has vertical asymptotes at $x = \sqrt{2}$ and $x = -\sqrt{2}$.

To find the horizontal asymptotes, we let $x \to \infty$ and $x \to -\infty$. Factoring out the leading term x^2 from both numerator and denominator, we find

$$\lim_{x\to\infty} \frac{2x^2 + x + 1}{x^2 - 2} = \lim_{x\to\infty} \frac{x^2}{x^2} \cdot \frac{2 + (1/x) + (1/x^2)}{1 - (2/x^2)} = 2$$

So there is a horizontal asymptote and $y = 2$ as $x \to \infty$. A similar calculation shows that there is a horizontal asymptote at $y = 2$ as $x \to -\infty$.

(b), (6 points) Does the following function have any symmetry? If so, what kind of symmetry does it have?

$$f(x) = \frac{\sin(x)}{2 + \cos(x)} - x^3$$

Answer:

(b) Plugging in $-x$, we find that

$$f(-x) = \frac{\sin(-x)}{2 + \cos(-x)} - (-x)^3$$

Note that $\sin(x)$ and x^3 are odd functions, but $\cos(x)$ is even. Therefore,

$$f(-x) = \frac{-\sin(x)}{2 + \cos(x)} + x^3 = -\frac{\sin(x)}{2 + \cos(x)} + x^3 = -f(x)$$

and therefore the function is symmetric with odd symmetry.

(c), (6 points) Find the intervals of increase and decrease for the following function. Then find the points x where the function has a local maximum or local minimum.

$$f(x) = \frac{x^5}{5} - \frac{4x^3}{3} + \frac{7}{2}$$

Answer:

(c) Note that

$$f'(x) = x^4 - 4x^2 = x^2(x-2)(x+2)$$

The critical points are at $x = -2, 0, 2$. These divide the x-axis into four intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$ and $(2, \infty)$. Checking the sign of $f'(x)$ at representative points on these intervals, we find that $(-\infty, -2)$ and $(2, \infty)$ are intervals of increase, while $(-2, 0)$ and $(0, 2)$ are intervals of decrease. The first derivative test then tells us that $f(x)$ has a local maximum at $x = -2$ and a local minimum at $x = 2$.

(d), (6 points) Using the same function $f(x)$ as in part (c), find the intervals on which the function is concave up and concave down, and find the points of inflection.

Answer:

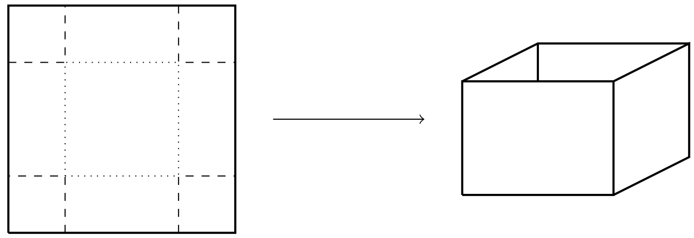
(d) Note that

$$f''(x) = 4x^3 - 8x = 4x(x^2 - 2) = 4x(x - \sqrt{2})(x + \sqrt{2})$$

So $f''(x) = 0$ at $x = -\sqrt{2}$, $x = 0$, and $x = \sqrt{2}$. These points divide the x-axis into the intervals $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$, and $(\sqrt{2}, \infty)$. Checking the sign of $f''(x)$ at representative points on these intervals, we find that $f''(x) > 0$ on $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$, so f is concave up on these intervals. Also, $f''(x) < 0$ on $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$, so f is concave down on these intervals. Since the concavity changes sign at $x = -\sqrt{2}$, $x = 0$, and at $x = \sqrt{2}$, all three of these points are points of inflection.

2. (21 points)

Suppose a box with an open top is to be made by cutting squares out of the corners of a 12 foot by 12 foot square piece of cardboard, then folding up the flaps to make sides. What is the maximum volume of such a box?



Answer:

Let x denote the side length of the square that is cut out. The volume is then $V(x) = x(12-2x)^2 = 4x^3 - 48x^2 + 144x$. Taking the derivative, $V'(x) = 12x^2 - 96x + 144 = 12(x^2 - 8x + 12) = 12(x-2)(x-6)$. We can see from the first derivative test that $x = 2$ is a local max, and is also a global max since at the endpoints $x = 0, 6$, the volume is 0. Thus the maximum volume is $V(2) = 2(12-2(2))^2 = 2(8)^2 = 128$ cubic feet.

3. (15 points) Find the antiderivatives of the following functions.

(a), (5 points)

$$\frac{x^3 - 4x}{x^{3/2}}, \qquad \text{for } x > 0.$$

Answer:

(a) First we simplify.

$$\frac{x^3 - 4x}{x^{3/2}} = x^{3/2} - 4x^{-1/2}$$

Then we compute the antiderivative:

$$\frac{2}{5}x^{5/2} - 8x^{1/2} + C$$

(b), (5 points)

$$2\sin(x) - x^2$$

Answer:

(b)

$$-2\cos(x) - \frac{x^3}{3} + C$$

(c), (5 points)

$$2e^{x/2}$$

Answer:

(c)

$$4e^{x/2} + C$$

4. (20 points)

Evaluate the following definite integrals:

(a), (10 points)

$$\int_0^3 \sqrt{9 - x^2} dx$$

Answer:

(a) This is a quarter circle with radius 3, so it has area $\pi(3)^2/4 = 9\pi/4$.

(b), (10 points)

$$\int_{-2}^{1} \Big(|x| - 1\Big) dx$$

Answer:

(b) The graph consists of a triangle above the x-axis with base and height 1, and a triangle below the x-axis with base 2 and height 1, thus the integral is $\frac{1}{2}(1)(1) - \frac{1}{2}(2)(1) = -\frac{1}{2}$

5. (20 points) Consider the integral

$$\int_1^3 e^{\sqrt{x}} dx$$

Write a Riemann sum for this integral. Assume that the partition has $n = 4$ subintervals of equal length, and the points x_i^* are at the midpoint of each interval. Just write down the Riemann sum, and do not try to evaluate the integral.

Answer:

The interval of integration is $[1, 3]$, and the subintervals are $[1, 3/2], [3/2, 2], [2, 5/2], [5/2, 3]$. The length of each subinterval is $\Delta x = 1/2$, and the midpoints are $5/4, 7/4, 9/4, 11/4$.

Therefore the Riemann sum is

$$\sum_{i=1}^{4} f(x_i^*)\Delta x = e^{\sqrt{5/4}} \cdot (1/2) + e^{\sqrt{7/4}} \cdot (1/2) + e^{\sqrt{9/4}} \cdot (1/2) + e^{\sqrt{11/4}} \cdot (1/2)$$