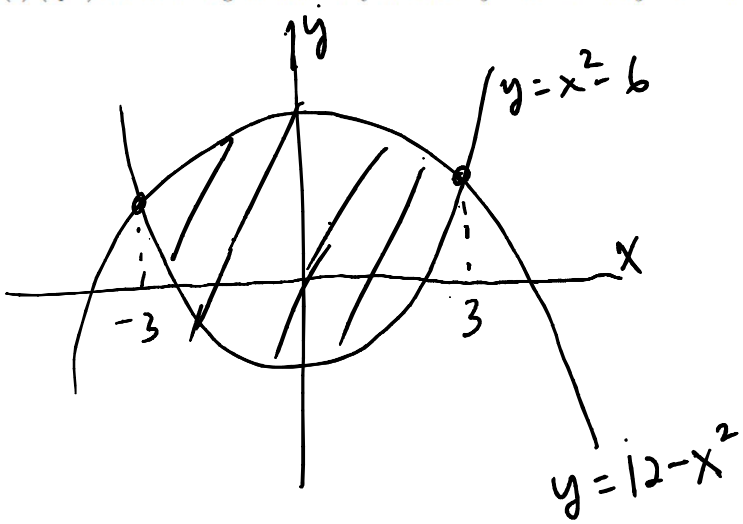
1. (18 points) Answer each part below.

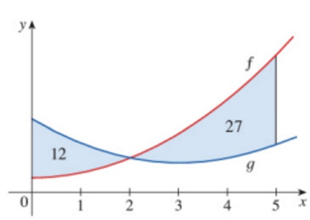
(a) (9pts) Sketch the region enclosed by the curves $y = 12 - x^2$ and $y = x^2 - 6$. Find its area.



$$12 - x^2 = x^2 - 6$$
$$\Rightarrow 2x^2 = 18$$
$$\Rightarrow x^2 = 9$$
$$\Rightarrow x = \pm 3$$

$$A = \int_{-3}^{3} \left[12 - x^2 - (x^2 - 6)\right] dx = 2\int_{-3}^{3} (x^2 + 9)\, dx$$
$$= 4 \int_0^3 (x^2 + 9)\, dx$$
$$= 4 \left(\frac{x^3}{3} + 9x \right) \Big|_0^3$$
$$= 4(9 + 27) = \boxed{144}$$

(b) (3pts) The graphs of two functions are shown with the area of the regions between the curves indicated. What is the value of $\int_0^5 (f(x) - g(x))\, dx$?



$$\rightarrow 12 = \int_0^2 (g - f)\, dx$$

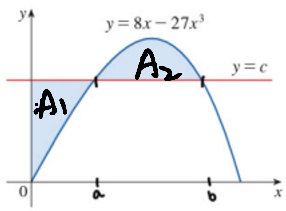
$$\Rightarrow \int_0^2 (f - g)\, dx = -12$$

Then, $\int_0^5 (f - g)\, dx = \int_0^2 (f - g)\, dx + \int_2^5 (f - g)\, dx$

$$= -12 + 27 = \boxed{15}$$

$$= -12+27 = \boxed{15}$$

(c) (6pts) The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the area of the shaded regions are equal. Express your answer as $c = f(b)$.



→ From part (b) we can see the areas A_1, A_2 will be equal if and only if

$$\int_0^b (f-g)\,dx = -A_1 + A_2 = 0,$$

where $f(x) = 8x - 27x^3$

$g(x) = c$.

Note: since the curves must intersect at $x = b \Rightarrow \boxed{8b - 27b^3 = c}$

Then, $$\int_0^b (8x - 27x^3 - c)\,dx = 0$$

$$\Rightarrow \left(4x^2 - \frac{27}{4}x^4 - cx\right)\Big|_0^b = 0$$

$$\Rightarrow 4b^2 - \frac{27}{4}b^4 - cb = 0$$

$$\Rightarrow \boxed{c = 4b - \frac{27}{4}b^3}$$

$$\Rightarrow \boxed{c = 4b - \frac{27}{4}b^3}$$

Combining w/ $c = 8b - 27b^3$ from above,

we get $4b - \frac{27}{4}b^3 = 8b - 27b^3$

$$\Rightarrow 0 = 4b - \frac{81}{4}b^3 = \frac{1}{4}b(16 - 81b^2)$$

$\Rightarrow b = 0, \pm 4/9$. Clearly $b > 0 \Rightarrow \boxed{b = \frac{4}{9}}$

Then, $c = f(b) = 8\left(\frac{4}{9}\right) - 27\left(\frac{4}{9}\right)^3$

2. (16 points) Use u-substitution and/or integration by parts to compute the following indefinite integrals:

(a) $\int x^2\sqrt{1+x}\,dx$

$u = 1+x$

$du = dx$

$x = u-1$

$\Rightarrow x^2 = (u-1)^2$

$$\int x^2\sqrt{1+x}\,dx = \int (u-1)^2\sqrt{u}\,du$$
$$= \int (u^2 - 2u + 1)\,u^{1/2}\,du$$
$$= \int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right) du$$
$$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C$$

(b) $\int (x - x^2)e^{-x}\,dx$

$$\int (x-x^2)e^{-x}\,dx = -e^{-x}(x-x^2) - \int (-e^{-x})(1-2x)\,dx$$

$u = x - x^2$ $\quad dv = e^{-x}dx$

$du = (1-2x)dx$ $\quad v = -e^{-x}$

$$= e^{-x}(x^2 - x) + \int (1-2x)e^{-x}\,dx$$

$u = x - x^2 \qquad dv = e^{-x}\,dx$

$du = (1-2x)\,dx \qquad v = -e^{-x}$

$u = 1-2x \qquad dv = e^{-x}\,dx$

$du = -2\,dx \qquad v = -e^{-x}$

$$= e^{-x}(x^2-x) + \int (1-2x)e^{-x}\,dx$$

$$= e^{-x}(x^2-x) - e^{-x}(1-2x) - \int (-e^{-x})(-2)\,dx$$

$$= \boxed{e^{-x}(x^2-x) + e^{-x}(2x-1) + 2e^{-x} + C}$$

3. (16 points) Use u-substitution and/or integration by parts to compute the following definite integrals:

(a) $\int_1^e \frac{\ln x}{x^2}\,dx$

$$\int_1^e \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x}\Big|_1^e + \int_1^e \frac{1}{x}\cdot\frac{1}{x}\,dx$$

$u = \ln x \qquad dv = x^{-2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -x^{-1}$

$$= -\frac{\ln(e)}{e} + \frac{\ln(1)}{1} - \frac{1}{x}\Big|_1^e$$

$$= -\frac{1}{e} - \frac{1}{e} + 1 = \boxed{1 - \frac{2}{e}}$$

(b) $\int_0^8 \frac{\sin(\sqrt[3]{x})}{\sqrt[3]{x^2}}\,dx$

$$\int_0^8 \frac{\sin(x^{1/3})}{x^{2/3}}\,dx = 3\int_{u=0}^2 \sin(u)\,du = -3\cos u\Big|_0^2$$

$$= -3(\cos(2) - \cos(0))$$

$$= \boxed{3 - 3\cos(2)}$$

$u = x^{1/3}$

$du = \frac{1}{3}x^{-2/3}\,dx$

$3\,du = \frac{dx}{x^{2/3}}$

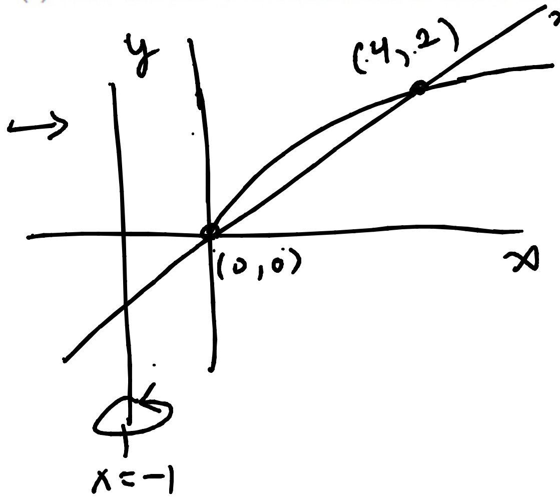
$x=0$: $u = 0^{1/3} = 0$

$x=8$: $u = 8^{1/3} = 2$

4. (24 points) Consider the solid given by rotating the region between $y = \sqrt{x}$ and $x = 2y$ about the line $x = -1$.

Note: you do not need to simplify to get full credit, but doing so may help you check your answers are correct (as parts (b) and (c) should yield the same number).

(a) Draw the area to be rotated and its axis of rotation.



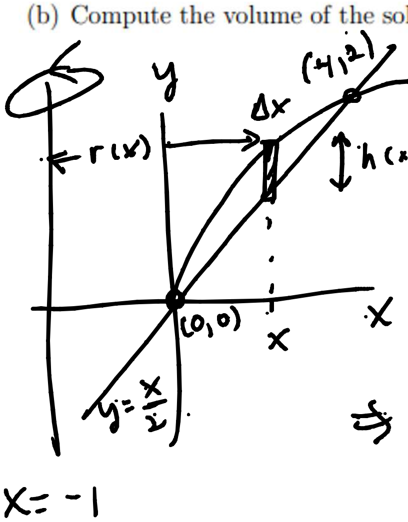
$y = \sqrt{x} \longleftrightarrow x = y^2$

$2y = y^2$

$\Rightarrow y(y-2) = 0$

$y = 0, 2$

(b) Compute the volume of the solid using the method of cylindrical shells.



$y = \sqrt{x}$

$r(x) = x - (-1) = x + 1$

$h(x) = \sqrt{x} - \frac{x}{2}$

$V_{shell}(x) \approx 2\pi\, r(x)\, h(x)\, \Delta x$

$$\Rightarrow V = 2\pi \int_0^4 (x+1)\left(\sqrt{x} - \frac{x}{2}\right) dx$$

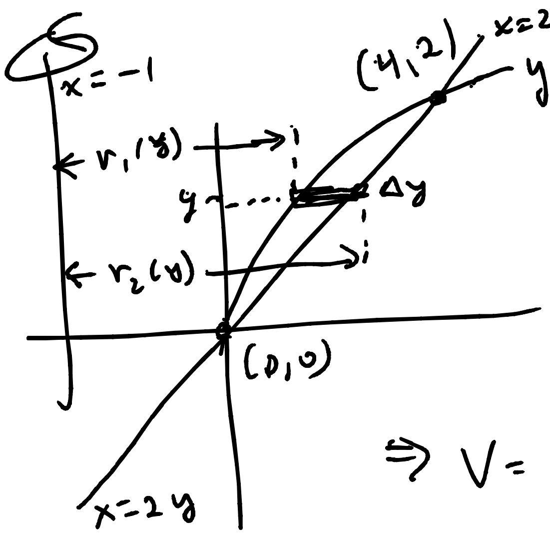
$$= 2\pi \int_0^4 \left(x^{3/2} - \frac{x^2}{2} + x^{1/2} - \frac{x}{2}\right) dx$$

$$= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^3}{6} + \frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]\Bigg|_0^4$$

$$= 2\pi \left[\frac{2}{5}(\sqrt{4})^5 - \frac{64}{6} + \frac{2}{3}(\sqrt{4})^3 - 4\right]$$

$$= 2\pi \left[\frac{64}{5} - \frac{64}{6} + \frac{16}{3} - 4\right] = \boxed{\frac{104\pi}{15}}$$

(c) Compute the volume of the solid using the method of washers.



$y=\sqrt{x} \Leftrightarrow x = y^2$

$$r_2(y) = 2y - (-1) = 2y+1$$

$$r_1(y) = y^2 - (-1) = y^2+1$$

$$V_{WASHER}(y) = \pi\left(r_2(y)^2 - r_1(y)^2\right)\Delta y$$

$$\Rightarrow V = \int_0^2 \pi\left[(2y+1)^2 - (y^2+1)^2\right] dy$$

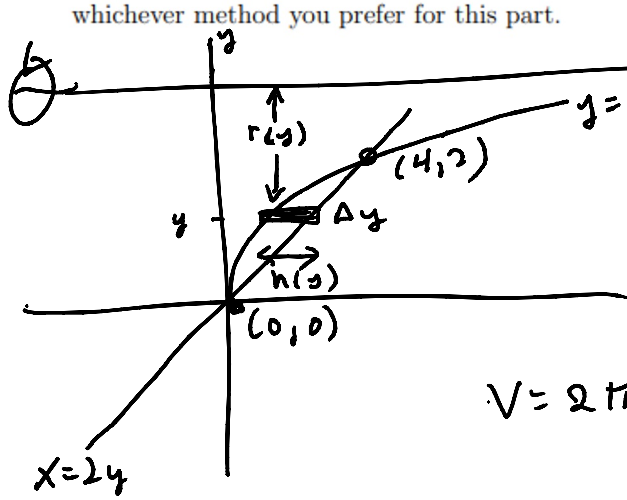
$$= \pi \int_0^2 (4y^2+4y+1-y^4-2y^2-1)\, dy$$

$$= \pi \int_0^2 (-y^4+2y^2+4y)\, dy$$

$$= \pi\left(-\frac{1}{5}y^5 + \frac{2}{3}y^3 + 2y^2\right)\Big|_0^2$$

$$= \pi\left(-\frac{32}{5} + \frac{16}{3} + 8\right) = \boxed{\frac{104\pi}{15}}$$

(d) Find the volume of the solid given by rotating the same region about the line $y = 5$. You may use whichever method you prefer for this part.



$y=\sqrt{x} \Leftrightarrow x = y^2$

$$V_{shell}(y) \approx 2\pi\, r(y)\, h(y)\, \Delta y$$

$$= 2\pi(5-y)(2y-y^2)\Delta y$$

$$V = 2\pi \int_0^2 (5-y)(2y-y^2)\, dy$$

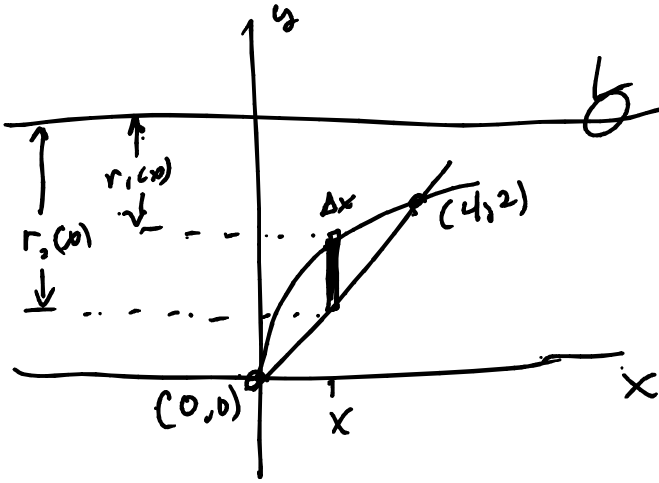
$$= 2\pi \int_0^2 (10y - 5y^2 - 2y^2 + y^3)\, dy$$

$$= 2\pi \int_0^2 (10y - 7y^2 + y^3)\,dy$$

$$= 2\pi \left(5y^2 - \frac{7}{3}y^3 + \frac{y^4}{4}\right)\Big|_0^2$$

$$= 2\pi \left(20 - \frac{56}{3} + 4\right) = \boxed{\frac{32\pi}{3}}$$

<u>OR</u> using WASHER:



$$V_{WASH}(x) = \pi\left(r_2(x)^2 - r_1^2(x)\right)\Delta x$$

$$V = \pi \int_0^4 \left[\left(5 - \frac{x}{2}\right)^2 - \left(5 - \sqrt{x}\right)^2\right] dx$$

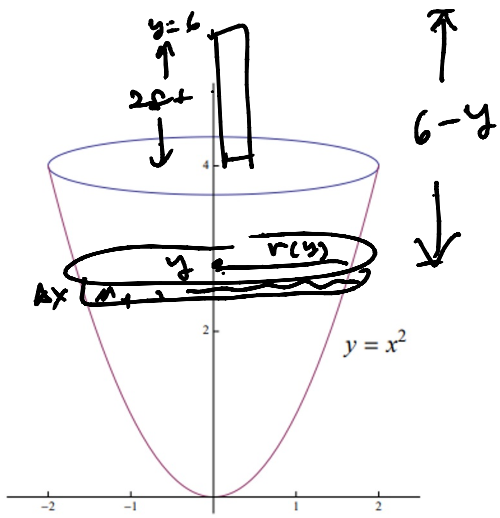
$$= \pi \int_0^4 \left[25 - 5x + \frac{x^2}{4} - 25 + 10\sqrt{x} - x\right] dx$$

$$= \pi \int_0^4 \left(\frac{x^2}{4} - 6x + 10x^{1/2}\right) dx$$

$$= \pi \left(\frac{4^3}{12} - \frac{6}{2}(4)^2 + \frac{20}{3}(4)^{3/2}\right)$$

$$= \pi \left(\frac{16}{3} - 48 + \frac{160}{3}\right) = \boxed{\frac{32\pi}{3}}$$

5. (12 points) Consider a tank generated by rotating the region bounded by the y-axis, $y = x^2$, and $y = 4$, about the y-axis (units are in feet). Suppose the tank is completely filled with a fluid that weighs $100\text{lbs}/\text{ft}^3$. Find the work done to empty the tank by pumping the water through a pipe 2 ft above the tank.



Slice at y is a disk of radius $r(y) = \sqrt{y}$

$\Rightarrow V_{slice}(y) = \pi(\sqrt{y})^2 \Delta y$

$= \pi y \Delta y \text{ ft}^3$

$F_{slice} = (100 \text{ lbs}/\text{ft}^3)(\pi y \Delta y \text{ ft}^3)$

$= 100\pi y \Delta y$ lbs.

$D_{slice}(y) \approx 6 - y$

$\Rightarrow W_{slice}(y) \approx 100\pi y(6-y)\Delta y$ ft-lbs.

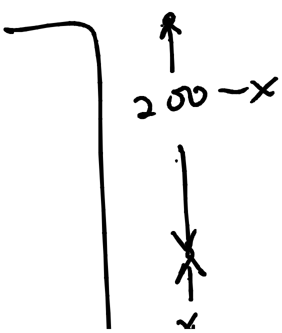
$$\Rightarrow W = 100\pi \int_0^4 y(6-y)\,dy$$

$$= 100\pi\left(3y^2 - \frac{y^3}{3}\right)\Big|_0^4 = \boxed{8000\pi \text{ ft-lbs}}$$

6. (14 points) Answer each part below.

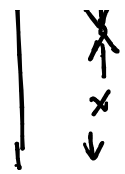
(a) Suppose a 200 ft cable weighing 800 lbs is hanging over the side of a tall building. **SET-UP ONLY** an integral to find the work done in towing the cable **halfway** to the top. Note: *Set-up only* means your answer should be an integral expression, but you should not try to evaluate the integral.

After x has been lifted up, there is $200 - x$ ft left hanging.



$\text{Density} = \frac{800}{200} = 4 \text{ lbs}/\text{ft}$

$F(x) = 4(200 - x)$ lbs.



$F(x) = 4(200 - x)$ lbs.

$$\boxed{W = 4\int_0^{100} (200 - x)\,dx \text{ ft-lbs}}$$

(b) It took 1800 Joules of work to stretch a spring from its natural length of 2 m to a length of 5 m. How much work is done in stretching the spring from length 4 m to length 10m?

$$k \int_0^3 x\,dx = 1800$$

$$\Rightarrow \frac{k}{2}(3)^2 = 1800 \Rightarrow \boxed{k = 400 \text{ N/m}}$$

4m $\Rightarrow$ $x = 2$ m beyond nat. length

10 m $\Rightarrow$ $x = 8$ m beyond

$$W = \int_2^8 400x\,dx = 200x^2\Big|_2^8$$

$$= 200(64 - 4) = 12000 \text{ J}$$