1. (14 points) Consider a particle moving on the real line whose acceleration at time t is given as $a(t) = 2t + t\sqrt{t}$. Also, assume the initial velocity and position of the particle at time $t = 0$ are given by $v(0) = 1$ and $s(0) = 0$, respectively. Find the velocity and position of the particle at $t = 1$.

$\rightarrow \quad a(t) = 2t + t^{3/2}$

$\Rightarrow \quad v(t) = t^2 + \frac{2}{5}t^{5/2} + C_1$

$v(0) = 1 \Rightarrow \quad 0^2 + \frac{2}{5}0^{5/2} + C_1 = 1$

$\Rightarrow \quad C_1 = 1$

So, $\boxed{v(t) = t^2 + \frac{2}{5}t^{5/2} + 1}$

$\Rightarrow \quad s(t) = \frac{t^3}{3} + \frac{2}{5}\cdot\frac{2}{7}t^{7/2} + t + C_2$

$s(0) = 0 \Rightarrow \quad C_1 = 0$

So, $\boxed{s(t) = \frac{t^3}{3} + \frac{4}{35}t^{7/2} + t}$

2. (16 points) Compute each the following integrals:

(a) $\displaystyle\int_0^{\pi/4} \cos(x) - \frac{1}{\cos^2(x)}\, dx$

$= \displaystyle\int_0^{\pi/4} (\cos x - \sec^2 x)\, dx$

$= \sin x \Big|_0^{\pi/4} - \tan x \Big|_0^{\pi/4}$

$= \sin(\pi/4) - \sin(0) - (\tan(\pi/4) - \tan(0))$

$= \frac{\sqrt{2}}{2} - 0 - (1 - 0)$

$= \boxed{\frac{\sqrt{2}}{2} - 1}$

(b) $\displaystyle\int \frac{2x^2 + \sqrt{x} - 1}{x} + 2e^x \, dx$

$= \int \left(\frac{2x^2}{x} + \frac{x^{1/2}}{x} - \frac{1}{x}\right) dx$

$= \int (2x + x^{-1/2} - x^{-1}) \, dx$

$= \boxed{x^2 + 2x^{1/2} - \ln|x| + C}$

3. (14 points) Find the right Riemann sum for n-subintervals and take the limit as $n \to \infty$ to evaluate the definite integral

$$\int_0^2 3x \, dx.$$

$\rightarrow \Delta x = \frac{2-0}{n} = 2/n$, $x_i = a + i\Delta x$

$= 0 + i\,2/n$

$= 2i/n$

$\int_0^2 3x \, dx = \lim_{n\to\infty} R_n$, where

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \sum_{i=1}^{n} f(2i/n)\cdot\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n} 3(2i/n)$$

$$= \frac{12}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{12}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= 6\left(\frac{n+1}{n}\right)$$

Then $\displaystyle\int_0^2 3x\,dx = \lim_{n\to\infty} 6\left(\frac{n+1}{n}\right)$

$$= 6\lim_{n\to\infty}\left(1+\cancel{\tfrac{1}{n}}^{\,0}\right)$$

$$= \boxed{6}$$

4. (14 points) Answer each part below.

(a) If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)\,dx = 16$, find $\int_0^9 [2f(x) + 3g(x)]\,dx$.

$$= 2\int_0^9 f(x)dx + 3\int_0^9 g(x)dx$$

$$= 2(37) + 3(16)$$

$$= 122$$

$= 122$

(b) Find the derivative of $h(x) = \int_{\cos x}^{1} \frac{1}{1+t^6} dt$.

$\rightarrow \quad h(x) = -\int_1^{\cos x} \frac{dt}{1+t^6}$

let $y = -\int_1^u \frac{1}{1+t^6} dt$, where $u = \cos x$

$\Rightarrow \frac{du}{dx} = -\sin x$

By FTC-I $\rightarrow \frac{dy}{du} = \frac{-1}{1+u^6}$

$\Rightarrow h'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{-1}{1+u^6}(-\sin x)$

$= \boxed{\frac{\sin x}{1+\cos^6 x}}$

5. (26 points) Consider the function $f(x) = \frac{2x}{x^2-7}$.

(a) Find the domain of f.

f is defined everywhere except $x^2-7=0$

$\Rightarrow x = \pm\sqrt{7}$.

$\text{dom} f = (-\infty, -\sqrt{7}) \cup (-\sqrt{7}, \sqrt{7}) \cup (\sqrt{7}, \infty)$

(b) List all x and y intercepts of f.

x int: $\frac{2x}{x^2-7} = 0 \Rightarrow x = 0 \rightarrow$ this is $(0,0)$

y int: $y = f(0) = \frac{2(0)}{0^2-7} = 0 \rightarrow$ this is $(0,0)$

$(0,0)$ is the only intercept.

(c) Is f even, odd, or neither?

f is odd since $f(-x) = \frac{2(-x)}{(-x)^2-7}$

$= -\left(\frac{2x}{x^2-7}\right)$

$= -f(x)$

(d) Find any horizontal and vertical asymptotes of f or explain why none exist.

$\rightarrow$ vert. asym: f defined everywhere except $x = \pm\sqrt{7}$. Both are vertical asymptotes since:

$$\lim_{x \to -\sqrt{7}^-} \frac{2x}{x^2-7} = -\infty$$

b/c $\frac{2x}{x^2-7} \sim \frac{-2\sqrt{7}}{0} \sim \pm\infty$ when $x \to -\sqrt{7}^-$,

and $\frac{x}{x^2-7} = \frac{-}{+} < 0$.

similarly, $\lim_{x \to -\sqrt{7}^+} \frac{2x}{x^2-7} = +\infty$

$$\lim_{x \to \sqrt{7}^-} \frac{2x}{x^2-7} = -\infty$$

$$\lim_{x \to \sqrt{7}^+} \frac{2x}{x^2-7} = +\infty$$

(e) Find all critical numbers and where the function is increasing and where it is decreasing. For each critical number found determine whether it is a local maximum, minimum, or neither.

$$f(x) = \frac{2x}{x^2-7} \Rightarrow f'(x) = \frac{2\cdot(x^2-7) - 2x(2x)}{(x^2-7)^2}$$

$$= \frac{-2x^2-14}{(x^2-7)^2} = -2\,\frac{(x^2+7)}{(x^2-7)^2} \Big\} \text{ always } < 0$$

<u>Note</u>: $f'(x) < 0$ for all $x \neq \pm\sqrt{7}$.

So, f is decreasing on $(-\infty, -\sqrt{7}) \cup (-\sqrt{7}, \sqrt{7}) \cup (\sqrt{7}, \infty)$.

<u>Note</u>: no critical numbers on the domain of f.

($x = \pm\sqrt{7}$ are vertical asymptotes)

(f) Find where the function is concave up, and where it is concave down, and all x values at points of inflection.

$$f'(x) = -2\left\{\frac{x^2+7}{(x^2-7)^2}\right\} \Rightarrow f''(x) = -2\left\{\frac{2x(x^2-7)^2 - (x^2+7)\cdot 2(x^2-7)\cdot 2x}{(x^2-7)^4}\right\}$$

$$= \frac{-4x(x^2-7)}{(x^2-7)^4}\left[(x^2-7) - 2(x^2+7)\right]$$

$$= \frac{4x(x^2+7)}{(x^2-7)^3}$$

$f''(x)$ --- DNE +++ 0 - - - 0 ++++

$f''(x)$ — — — DNE + + + 0 — — — 0 + + + +

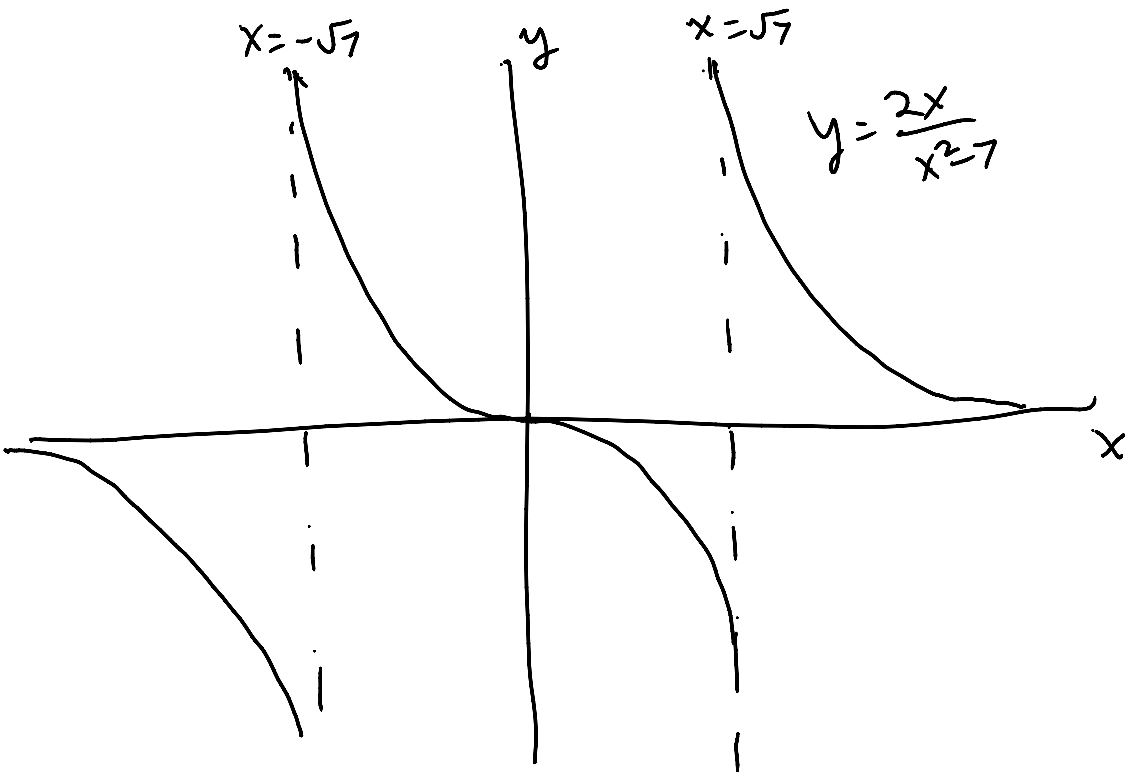
$-\sqrt{7}$ $\quad$ 0 $\quad$ $\sqrt{7}$

$\to$ f concave up on $(-\infty, -\sqrt{7}) \cup (0, \sqrt{7})$

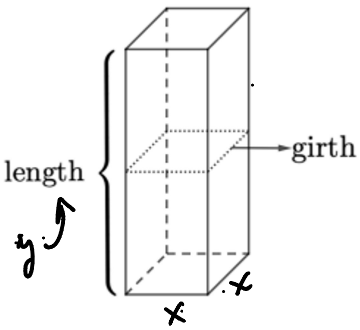
f concave down on $(-\sqrt{7}, 0) \cup (\sqrt{7}, \infty)$

Inflection when $x=0$ ($x = \pm\sqrt{7}$ are asymptotes)

(g) Use the information you found in the parts above to sketch the graph of $y = f(x)$.



6. (16 points) A package to be mailed using USPS may not measure more than 200*in* in length plus girth (see figure below). Find the dimensions of the rectangular box with square base of greatest volume that may be mailed.



→ Constraints: $y+4x\le 200$ → greatest volume $\Rightarrow y+4x=200$ $\Rightarrow \boxed{y=200-4x}$

Since $y\ge 0 \Rightarrow 4x\le 200$
$\Rightarrow \boxed{x\le 50}$

Goal: Max Volume $= x^2y = x^2(200-4x)$

Max. $V(x)=x^2(200-4x)$ on $0\le x\le 50$:

$V=200x^2-4x^3 \Rightarrow V'(x)=400x-12x^2$
$=4x(100-3x)$

$V'=0$: $x=0$ or $100-3x=0$
$\Rightarrow x=100/3$

The only critical number on $0<x<50$ is $x=100/3$. Compare w/ endpts. of $[0,50]$, largest must be max:

$V(0)=0^2(200-4(0))=0$
$V(50)=50^2(200-4(50))=0$
$V(\frac{100}{3})=(\frac{100}{3})^2(200-4(\frac{100}{3})$

$= 2\left(\frac{100}{3}\right)^3 \text{ in}^3$

so max. occurs when $x = \frac{100}{3}$ in

and $y = 200 - 4\left(\frac{100}{3}\right)$

$= \frac{200}{3}$ in.