# MTH 142

Final Exam

May 1, 2023

**Name:** SOLUTIONS

**UR ID:** ____________________

**Circle your Instructor's Name:**

Carissa Slone    Vanessa Matus de la Parra    Surena Hozoori    Mark Herman

**Instructions:**

• The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• When applicable, please put your final answer in the answer box. We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$.

• This exam is out of 100 points. You are responsible for checking that this exam has all 17 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____________________

| Part A | | |
|---|---|---|
| QUESTION | VALUE | SCORE |
| 1 | 24 | |
| 2 | 20 | |
| 3 | 24 | |
| 4 | 20 | |
| 5 | 12 | |
| TOTAL | 100 | |

| Part B | | |
|---|---|---|
| QUESTION | VALUE | SCORE |
| 1 | 24 | |
| 2 | 14 | |
| 3 | 14 | |
| 4 | 24 | |
| 5 | 14 | |
| 6 | 10 | |
| TOTAL | 100 | |

**Part A**

**1. (24 points)** This problem will guide you through graphing $f(x) = \dfrac{x}{x-1}$.

(a) Find the domain of $f$ as well as horizontal and vertical asymptotes.

domain$(f) = \{$where $x - 1 \neq 0\} = (-\infty, 1) \cup (1, +\infty)$

HA: $\displaystyle\lim_{x\to+\infty} \frac{x}{x-1} = \lim_{x\to+\infty} \frac{1}{1-1/x} = 1 = \lim_{x\to-\infty} \frac{x}{x-1}$

$\rightsquigarrow \boxed{y = 1 \ : \text{HA}}$

VA: $\displaystyle\lim_{x\to 1^+} \frac{x}{x-1} = +\infty \rightsquigarrow \boxed{x = 1 \ : \text{VA}}$

( $f(x)$ continuous everywhere else)

(b) List intervals where $f$ is increasing/decreasing.

$$f'(x) = \frac{1(x-1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$\Rightarrow$ $f'(x)$: $-$ $\quad$ $-$ $\longrightarrow x$

$x=1$

$f(x)$ is decreasing on $(-\infty, 1)$ and $(1, +\infty)$

——— nowhere increasing.

(c) List intervals where $f$ is concave up/down.

$$f''(x) = \frac{2}{(x-1)^3}$$

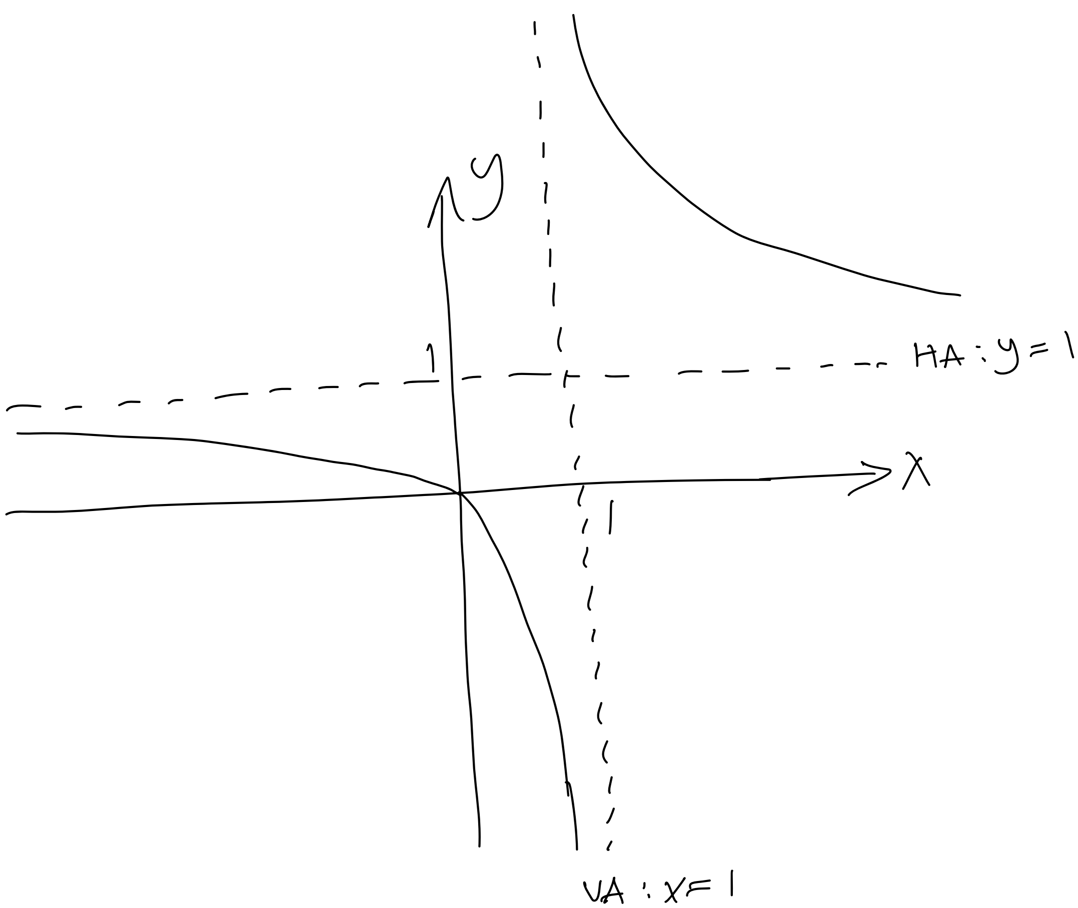
$f''(x)$ $\quad - \quad$ | $\quad + \quad$ $\to x$

$\qquad\qquad\qquad 1$

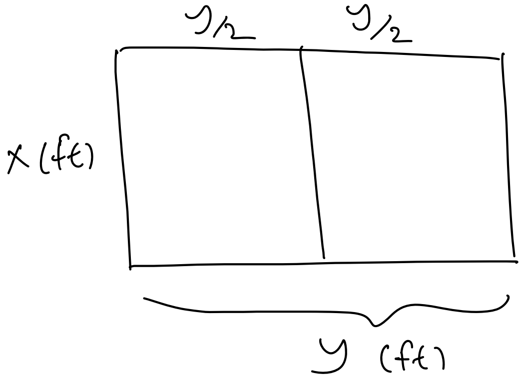
$f(x)$: concave up on $(1, +\infty)$

concave down on $(-\infty, 1)$

(d) Graph the function using the information above.



**2. (20 points)** A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?




fence used $= 2y + 3x$

area of the rectangle $= 1.5 \times 10^6 \text{ ft}^2 = xy$

$\Rightarrow y = \dfrac{1.5 \times 10^6}{x}$

$\Rightarrow$ fence used in terms of $x$ $= f(x) = 2\dfrac{1.5 \times 10^6}{x} + 3x$

$\rightsquigarrow$ in order to minimize the cost of fence, need to minimize $f(x)$ for $0 < x < \infty$

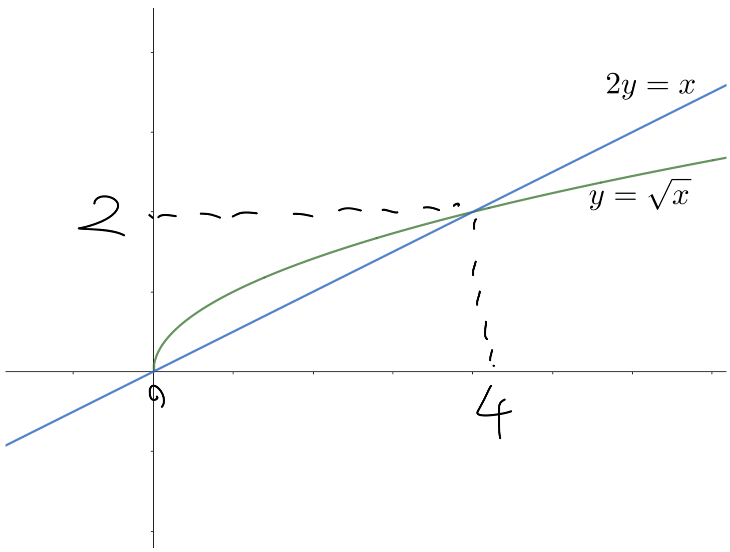
$\rightsquigarrow f'(x) = -\dfrac{3 \times 10^6}{x^2} + 3 = 0$

to find the critical pts over $(0, +\infty)$

$\dfrac{3 \times 10^6}{x^2} = 3 \Rightarrow \boxed{x = 10^3 \text{ ft}}$

$\rightsquigarrow \boxed{y = \dfrac{1.5 \times 10^6}{10^3} = 1.5 \times 10^3 \text{ ft}}$ $\leftarrow$ the dimensions to use to minimize the cost.

**3. (24 points)** This problem is concerning the region between $2y = x$ and $y = \sqrt{x}$ shown in the picture.




(a) Use the method of **washers** to *set up only* an integral for the volume of the solid obtained by rotating this area about the $x$-axis. *Set up only* means your answer should be an integral expression and you should NOT attempt evaluate the integral.

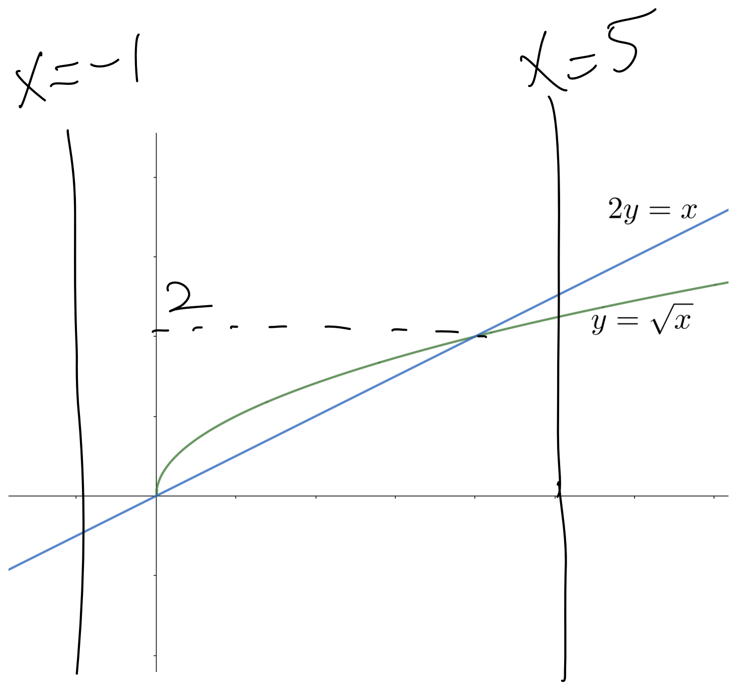
to find the interval of integration: $\begin{cases} 2y = x \\ y = \sqrt{x} \end{cases} \rightarrow \frac{x}{2} = \sqrt{x}$

$\rightarrow \begin{cases} x = 4, \; y = 2 \\ x = 0, \; y = 0 \end{cases}$

$$\text{Volume} = \int_0^4 \pi \left[ r_{out}^2(x) - r_{inn}^2(x) \right] dx$$

$$\begin{cases} r_{out}(x) = \sqrt{x} \\ r_{inn}(x) = x/2 \end{cases}$$

$$\text{Volume} = \int_0^4 \pi \left[ x - x^2/4 \right] dx$$




(b) Use the method of **washers** to *set up only* an integral for the volume of the solid obtained by rotating this area about the line $x = -1$.

$$\text{Volume} = \int_0^2 \pi \left(r_{out}^2(y) - r_{inn}^2(y)\right) dy$$

$$= \int_0^2 \pi \left[(1+2y)^2 - (1+y^2)^2\right] dy$$

$$\begin{cases} r_{out} = 1+2y \\ r_{inn} = 1+y^2 \end{cases}$$

(c) Use the method of **cylindrical shells** to *set up only* an integral for the volume of the solid obtained by rotating this area about the line $x = 5$.

$$\text{Volume} = \int_0^4 2\pi\, r(x) h(x)\, dx$$

$$= \int_0^4 2\pi (5-x)\left(\sqrt{x} - x/2\right) dx$$

$$\begin{cases} r = 5 - x \\ h = \sqrt{x} - x/2 \end{cases}$$

**4. (20 points)** Evaluate the following integrals:

(a) $\int_1^e \frac{(\ln x)^2}{x} dx$ $= \int_0^1 u^2 du = \left[\frac{1}{3} u^3\right]_0^1 = \boxed{\frac{1}{3}}$

u-sub: $\begin{cases} u = \ln(x) \\ du = \frac{1}{x} dx \\ u(1) = 0 \\ u(e) = 1 \end{cases}$

(b) $\int_0^1 x^2 e^{-x} dx$ $= \left[-x^2 e^{-x}\right]_0^1 - \int_0^1 -2x e^{-x} dx = \left[-x^2 e^{-x}\right]_0^1 - \left[2x e^{-x}\right]_0^1 + \int_0^1 2e^{-x} dx$

$= \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^1$

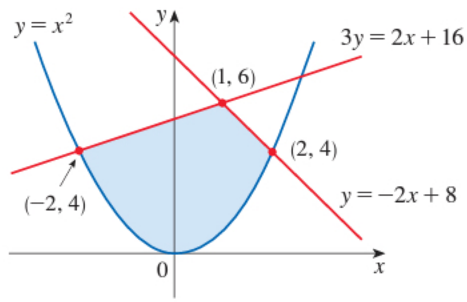
$= \left[\left(-e^{-1} - 2e^{-1} - 2e^{-1}\right) - (-2)\right]$

$= \boxed{-5e^{-1} + 2}$

IBP: $\begin{cases} u = x^2 \\ dv = e^{-x} dx \\ \hline du = 2x dx \\ v = -e^{-x} \end{cases}$

IBP $\begin{cases} u = 2x \\ dv = -e^{-x} dx \\ \hline du = 2 dx \\ v = e^{-x} \end{cases}$

**5. (12 points)** Find the area of the shaded region.




$$\text{Area} = \int_{-2}^{2} \text{upper}(x) - \text{lower}(x)\,dx = \int_{-2}^{1} \left(\frac{2}{3}x + \frac{16}{3}\right) - (x^2)\,dx + \int_{1}^{2} (-2x+8) - (x^2)\,dx$$

$$= \left[ x^2/3 + 16x/3 - x^3/3 \right]_{-2}^{1} + \left[ -x^2 + 8x - x^3/3 \right]_{1}^{2}$$

$$= (16/3) - \left(4/3 - 32/3 + 8/3\right) + \left(-4 + 16 - 8/3\right) - \left(-1 + 8 - 1/3\right)$$

**Part B**

**1. (24 points)** Evaluate the following integrals:

(a) $\int \sin^3(x)dx$ $= \int \sin^2 x \sin x \, dx = \int (1-\cos^2 x)\sin x \, dx = \int u^2 - 1 \, du$

$\begin{cases} u = \cos x \\ du = -\sin x \, dx \end{cases}$

$= u^3/3 - u + C = \cos^3 x/3 - \cos x + C$

(b) $\int \cos^2(2x)dx$ $= \int \frac{\cos(4x)+1}{2} dx = 1/8 \sin(4x) + x/2 + C$

**2. (14 points)** Evaluate the integral $\int \frac{x}{\sqrt{1-x^2}}\,dx$ using a trigonometric substitution. Note: no points will be awarded if you do not use a trig substitution. Your answer should be written in simplest form, not including an inverse trig function.

$$\begin{cases} x = \sin\theta \\ dx = \cos\theta\, d\theta \\ \sqrt{1-x^2} = \cos\theta \\ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \end{cases} \implies \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{\sin\theta}{\cos\theta}\cos\theta\, d\theta = \int \sin\theta\, d\theta$$

$$= -\cos\theta + C$$

$$= -\sqrt{1-x^2} + C$$

**3. (14 points)** Evaluate $\int \frac{x+1}{x^3+2x}\,dx$

$$x^3+2x = x\underbrace{(x^2+2)}_{\text{irreducible}} \implies \frac{x+1}{x^3+2x} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$$

$$\implies x+1 = A(x^2+2) + (Bx+C)x = (A+B)x^2 + (C)x + (2A)$$

$\underline{x=0} \rightsquigarrow 1 = 2A \to \boxed{A = 1/2}$

$0x^2 = (A+B)x^2 \xrightarrow{A+B=0} \boxed{B = -1/2}$

$1x = Cx \rightsquigarrow \boxed{C=1}$

$$\implies \int \frac{x+1}{x^3+2x}\,dx = \int \frac{1/2}{x} + \frac{-1/2\,x+1}{x^2+2}\,dx$$

$$= \int \frac{1/2}{x} + \underbrace{\frac{-1/2\,x}{x^2+2}}_{\text{u-sub}\,\left\{\begin{array}{l} u=x^2+2 \\ du=2x\,dx \end{array}\right.} + \frac{1}{x^2+2}\,dx = 1/2 \ln|x| - 1/4 \ln|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

**4. (24 points)** Determine the convergence of the following improper integrals. If it converges, find its value. Otherwise, show that it diverges.

(a) $\displaystyle\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dt + \lim_{t\to 1^+} \int_t^3 \frac{1}{x-1}\,dx$$

$$= \underbrace{\lim_{t\to 1^-} \left(\ln|t-1| - \cancel{\ln(1)}^{\,0}\right)}_{\text{divergent since } \lim_{t\to 1} \ln|t-1| = -\infty} + \underbrace{\lim_{t\to 1^+} \left(\ln(2) - \ln|t-1|\right)}$$

$$\boxed{\int_0^3 \frac{1}{x-1}\,dx \text{ : divergent}}$$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{+\infty} \frac{dx}{1+x^2}$

$$= \lim_{t\to -\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to +\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to -\infty} \left[\tan^{-1}(x)\right]_t^0 + \lim_{t\to +\infty} \left[\tan^{-1}(x)\right]_0^t$$

$$= \lim_{t\to -\infty} \left[\tan^{-1}(0) - \tan^{-1}(t)\right] + \lim_{t\to +\infty} \left[\tan^{-1}(t) - \tan^{-1}(0)\right]$$

$$= \left[(0) - (-\pi/2)\right] + \left[(\pi/2) - (0)\right] = \pi \quad \text{convergent}$$

**5. (14 points)** Use the Comparison Theorem to determine if the integral is convergent/divergent. Your answer should be clear as to how you are using the comparison theorem and why it applies. (Note: you do not need to find the value of the integrals, just determine convergence or divergence).

(a) $\int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}} dx$

by p-test $\int_1^\infty \frac{1}{\sqrt{x}} dx = \int_1^\infty \frac{1}{x^{1/2}} dx$ : divergent

$p = 1/2 < 1$

$\frac{1}{\sqrt{x}} < \frac{1+\sin^2 x}{\sqrt{x}} \xrightarrow{\text{comparison test}} \int_1^\infty \frac{1+\sin^2 x}{\sqrt{x}} dx$ : divergent

(b) $\int_5^\infty \frac{x}{x^3+1} dx$

by p-test $\int_5^\infty \frac{1}{x^2} dx$ : convergent

$p = 2 > 1$

on $(5, +\infty)$ : $0 < \frac{x}{x^3+1} < \frac{x}{x^3} = 1/x^2 \xrightarrow{\text{comparison test}} \int_5^{+\infty} \frac{x}{x^3+1} dx$ : convergent

**6. (10 points)**

Find the arc length of the curve

$$y = 1 + 6x^{3/2}, \quad 0 \le x \le 1.$$

$$dy/dx = 9x^{1/2}$$

$$\Rightarrow \text{arc length} = \int_0^1 \sqrt{1+(dy/dx)^2}\,dx = \int_0^1 \sqrt{1+81x}\,dx$$

$$= \int_1^{82} \frac{1}{81} u^{1/2}\,du = \left[\frac{2}{3\cdot(81)} u^{3/2}\right]_1^{82} = \boxed{\frac{2}{243}(82)^{3/2} - \frac{2}{243}}$$

u-sub $\begin{cases} u = 1+81x \\ du = 81\,dx \\ u(0) = 1 \\ u(1) = 82 \end{cases}$

**EXTRA PAGE.** You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.

### Sum Formulas

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

### Volume Formulas

- Sphere: $V = \frac{4\pi}{3}r^3$
- Circular cone: $V = \pi r^2 \frac{h}{3}$
- Cylinder: $V = \pi r^2 h$
- Cube: $V = s^3$
- Rectangular prism (box): $V = lwh$

### Trig Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
- $\sin(a)\cos(b) = \dfrac{\sin(a-b)+\sin(a+b)}{2}$
- $\sin(a)\sin(b) = \dfrac{\cos(a-b)-\cos(a+b)}{2}$
- $\cos(a)\cos(b) = \dfrac{\cos(a-b)+\cos(a+b)}{2}$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$
- $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

### More Formulas

- $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
- $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x| + C$
- $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(x/a) + C$
- $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(x/a) + C$
- Arc Length: $L = \int_a^b \sqrt{1+(dy/dx)^2}\,dx$ or $L = \int_c^d \sqrt{1+(dx/dy)^2}\,dy$