Basic assumption $\lim_{x\to a} f(x)$ $\lim_{x\to a} g(x)$ exist

$f(x) = \frac{1}{x}$ $\lim_{x\to 0} f(x) = \infty$ $\lim_{x\to 0} f(x)\cdot g(x) = \lim_{x\to 0} 1 = 1$

$(0, +\infty)$

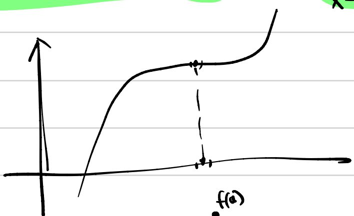
$g(x) = x$ $\lim_{x\to 0} g(x) = 0$

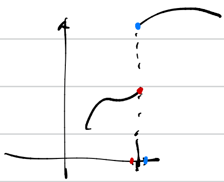
$\cancel{1 = 0 \times \infty}$? Not true

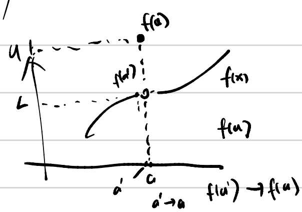
① if f is a polynomial or a rational function a in domain of f

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

if f cts at a $\Leftrightarrow$ $\lim_{x\to a} f(x) = f(a)$





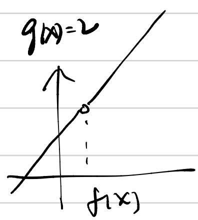


$\forall x \neq 1$ $x = 1$

$f(x) = g(x)$

$g(x) = x+1$

$g(x) = 2$



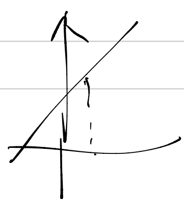
$f(x)$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x\to 1} x+1 = 1+1 = 2$$

x approaches to 1

$x \neq 1 \Rightarrow x-1 \neq 0$ $x-1$

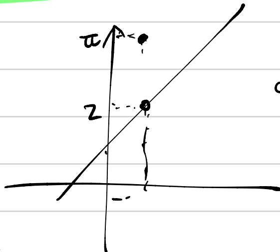
why we can cancel



if $f(x) = g(x)$ when $x \neq a$.

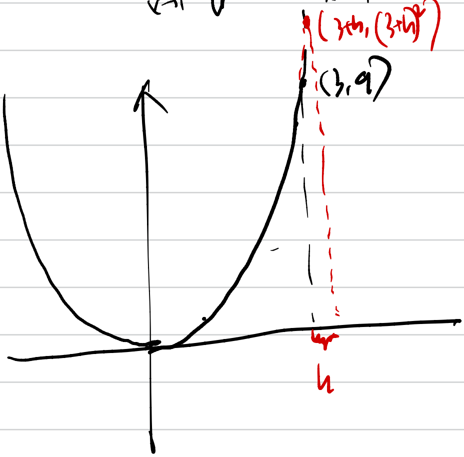
Then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$ provided the limits exist

$$g(x) = \begin{cases} x+1 & x \neq 1 \\ \pi & x = 1 \end{cases}$$



$g(1) = \pi$

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} x+1 = 2$$



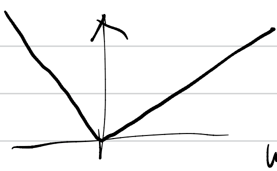
$$\lim_{x \to 3} \frac{x^2 - 9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} x+3 = 6$$

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} \frac{9+6h+h^2-9}{h}$$

$$= \lim_{h \to 0} \frac{6h+h^2}{h}$$

$$= \lim_{h \to 0} 6+h = 6+0 = 6$$

$f(x) = |x|$ $\quad$ $f(x)$ has limit on 0



$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

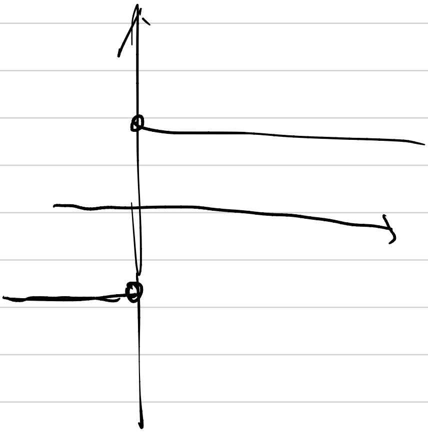
$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = -0 = 0$$

Left limit = Right limit

$0 = 0$

at $x = 0$

$\lim\limits_{x \to 0} |x|$ exist $= 0$



$f(x) = \frac{|x|}{x}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{|x|}{x}$

↙ Not able to use quotient law here

Limit doesn't exist

Right side limit

$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1$

Left side limit

$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$

$1 \neq -1$ limit doesn't exist at $x \to 0$

$$\boxed{f(x) = \begin{cases} \sqrt{x-4} & x > 4 \\ 8 - 2x & x < 4 \end{cases}}$$

Right side limit

$\lim_{x \to 4^+} \sqrt{x-4} = \sqrt{\lim_{x \to 4^+} x - 4}$

$= \sqrt{4-4}$

$= \sqrt{0} = 0$

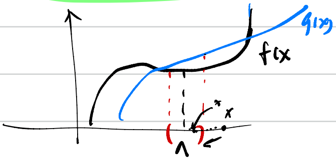
Left side lim

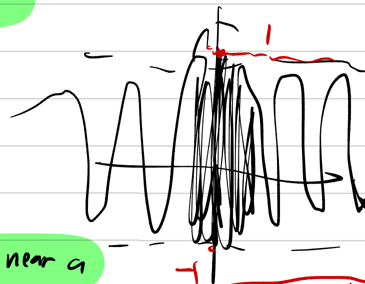
$\lim_{x \to 4^-} 8 - 2x = 8 - 2 \times 4 = 0$

$0 = 0$

$\Rightarrow \lim_{x \to 4}$ exist

$f(x) \le g(x)$ when x is near a, and the limit of f, g both exist as $x \to a$. then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$





$\lim_{x\to 0} \sin\frac{1}{x}$ × exist

Squeeze theorem

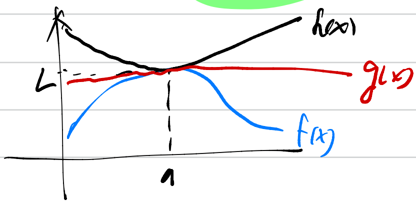
if $f(x) \le g(x) \le h(x)$ when x near a

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

Then $\lim_{x\to a} g(x) = L$

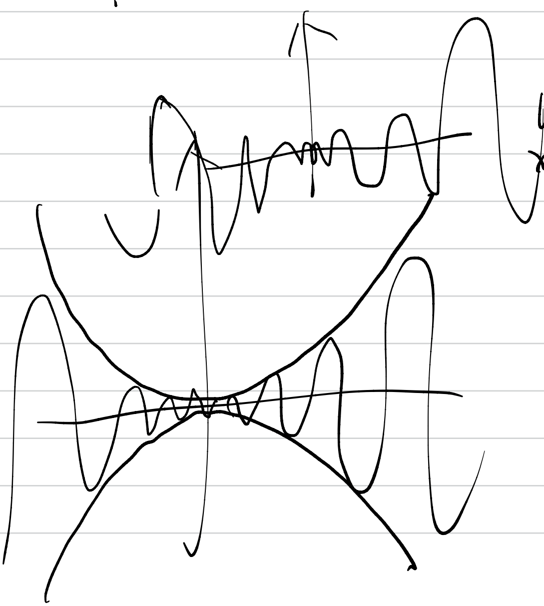
$\boxed{f(x) = x^2 \sin\frac{1}{x}}$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x}$?



$-1 \le \sin(\frac{1}{x}) \le 1$

$-x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$



$\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} -x^2 = 0$

$\lim f(x) = 0$

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

$f(x)$ defined somewhere near 1

$\sqrt{x} \le f(x) \le x^3$

$\lim_{x\to1} f(x)$ $\quad \lim_{x\to1} x^3 = 1 \quad \lim_{x\to1} \sqrt{x} = \sqrt{\lim_{x\to1} x} = \sqrt{1} = 1$

then $\lim_{x\to1} x^3 = \lim_{x\to1}\sqrt{x} \Rightarrow \lim_{x\to1} f(x) = 1$

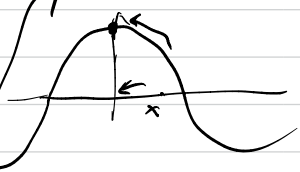
$\boxed{x \le f(x) \le x^2}$ $x\to0$

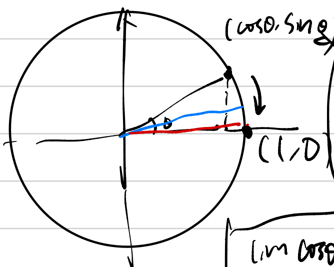
$\lim_{x\to0} x \le \lim_{x\to0} f(x) \le \lim_{x\to0} x^2$

$0 \le \lim_{x\to0} f(x) \le 0$

$\Rightarrow \lim_{x\to0} f(x) = 0$

$$\lim_{x\to0}\frac{\sin x+1}{\cos x+1} = \frac{\lim_{x\to0}\sin x+1}{\lim_{x\to0}\cos x+1} = \frac{0+1}{1+1} = \frac{1}{2}$$





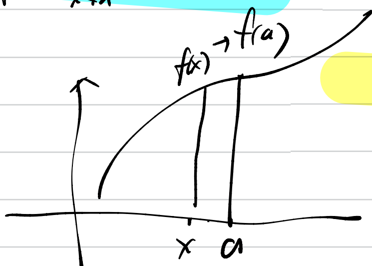
$\lim_{\theta\to0}\cos\theta = 1$

$\lim_{\theta\to0}\sin\theta = 0$

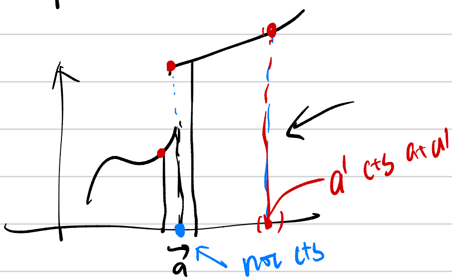
Continuity of a function

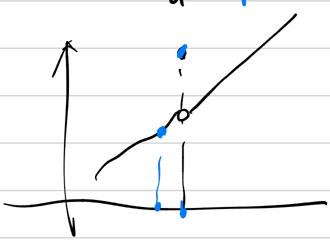
A function f is cts at a number a if
$\lim_{x \to a} f(x) = f(a)$

- $f(a)$ is defined (a is in domain of f)
- $\lim_{x \to a} f(x)$ exist
- $\lim_{x \to a} f(x) = f(a)$

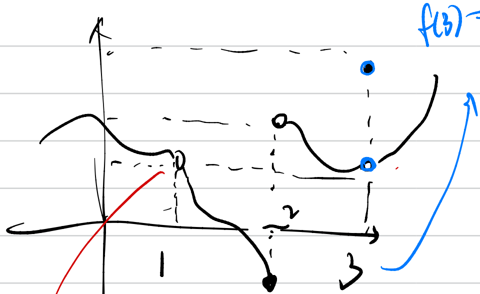


Small change in $x \Rightarrow$ small change in $f(x)$





$f(3) \neq \lim_{x \to 3} f(x)$



$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

at $x=2$. $\lim_{x \to 2} f(x)$ doesn't exist

$f(1)$ not defined

given $a \neq 2$

$$= \frac{(x-2)(x+1)}{x-2}$$

$$\boxed{\lim_{x\to a} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to a} x+1 = a+1 = f(a)}$$

(a) $f(x) = \dfrac{x^2-x-2}{x-2}$

$f(2)$ not defined

at point 2 $\quad \lim_{x\to 2} f(x) = f(2)$ not exist / not hold

discontinuous at $x=2$

$$f(a) = \frac{a^2-a-2}{a-2}$$

$$= \frac{(a-2)(a+1)}{a-2}$$

$$= a+1$$

(b) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & x=2 \end{cases}$

$f(2)$ define $=1$ $\quad \lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{(x+1)(x-2)}{x-2}$

Still discontinuity at $x=2$

$$= \lim_{x\to 2} x+1$$

$$= 3 \neq f(2)$$

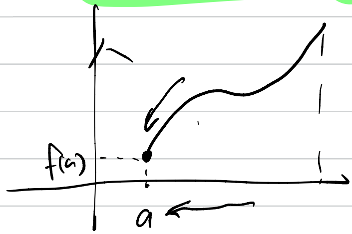
(c) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x=0 \end{cases}$

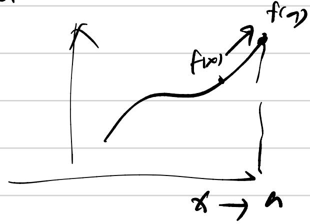
doesn't exist

$$\boxed{\lim_{x\to 0} f(x) = +\infty} \neq 1 = f(0)$$

function f is cts from right / left at number a

$\lim_{x \to a^+} f(x) = f(a)$ / $\lim_{x \to a^-} f(x) = f(a)$





A function is cts on an interval if it is cts at every point in the interval

cts on $[a, b]$

$f(x) = 1 - \sqrt{1 - x^2}$ WTS this cts $[-1, 1]$

$\lim_{x \to -1^+} f(x) = f(-1)$

$-1 < a < 1$ $\lim_{x \to a} f(x) = \lim_{x \to a} (1 - \sqrt{1 - x^2})$

$= 1 - \lim_{x \to a} \sqrt{1 - x^2}$

$= 1 - \sqrt{\lim_{x \to a} (1 - x^2)} = 1 - \sqrt{1 - a^2} = f(a)$

$\lim_{x \to 1^-} 1 - \sqrt{1 - x^2} = 1 - \sqrt{1 - 1}$

$= 1 = f(1)$

$f(a) = 1 - \sqrt{1 - a^2}$ f is cts at a if $-1 < a < 1$

$\left(\lim_{x\to a} f(x) = f(a),\ \lim_{x\to a} g(x) = g(a)\right)$

Theorem if f, g cts at a, c is a constant then following functions cts at a.

① $f+g$
② $f-g$
③ cf
④ fg
⑤ $\frac{f}{g}$ if $g(a) \neq 0$

$$\lim_{x\to a}(f+g) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$= f(a) + g(a) := (f+g)(a)$$

f, g cts on interval I

previous functions follows continuity on I

$p(x)$ polynomial $\lim_{x\to a} p(x) = p(a)$ direct substitution property

$p(x)$ is cts at a.

polynomial cts everywhere, that is, cts on $\mathbb{R}$: $(-\infty, \infty)$

$Q(x) = \frac{p(x)}{q(x)}$ if $a \in \mathrm{Dom}\,Q(x)$ $\lim_{x\to a} Q(x) = Q(a)$ $Q(x)$ is cts at a

$Q(x)$ is cts on $\mathrm{Dom}(Q(x))$

Theorem 7. following type of function are cts at every number in their domain

polynomial   Rational function   root function

trig function, inverse trig functions (by checking graph of function)

exponential function (made to be cts)   logarithm (inverse of cts function)

1-1

$\tan x = \frac{\sin x}{\cos x}$ a make $\cos a \neq 0$ ⇒ able to use quotient law

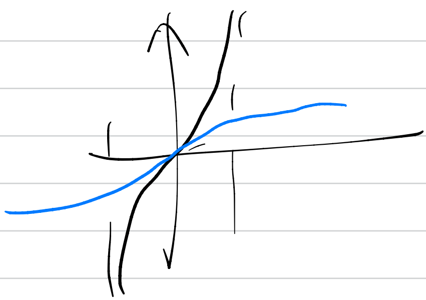
$$\lim_{x\to a} \tan x = \frac{\lim_{x\to a} \sin x}{\lim_{x\to a} \cos x} = \frac{\sin a}{\cos a} = \tan(a)$$

tan is cts on a

$\cos a = 0$ a is not in domain of $\tan x$

Thus, given $a \in$ Domain $\tan x$ ⇔ imply $\cos a \neq 0$ hence

$\lim_{x\to a} \tan$ exist $= \tan(a)$. ⇒ $\tan x$ cts at a



Find $\lim_{x \to -2} \frac{x^3+2x^2-1}{5-3x} = f(-2) = \frac{-1}{11}$

previous method

-2 in domain of f(x)

we can substitute $x=-2$

but f(x) is cts on its domain

hence cts on -2

by definition of cts

$2+\cos\pi = 2+(-1) = 1 \neq 0$

$\lim_{x \to \pi} \frac{\sin x}{2+\cos x} = \frac{\sin\pi}{2+\cos\pi}$

$f(x) = \sin x$ cts at π

$g(x) = 2+\cos x$ is cts at π

$= \frac{0}{1} = 0$

$\frac{f}{g}$ is cts at π