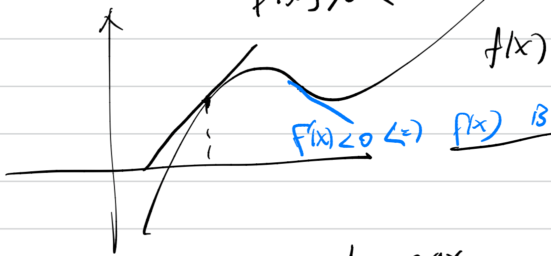
06/24

Increasing / Decreasing test

$f'(x) > 0 \Leftrightarrow f(x)$ is increasing at this point

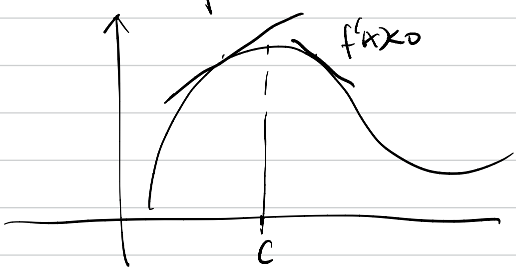


$f'(x) < 0 \Leftrightarrow f(x)$ is decreasing at this point

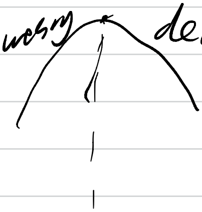
local max

$f'(x) > 0$

$f'(x) < 0$



local maxima

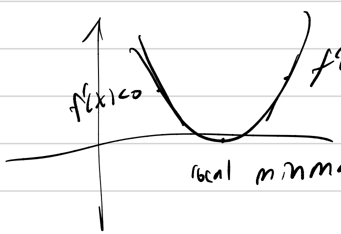


local min

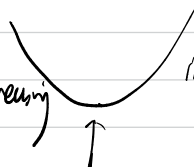
concave upward

$f'(x)$ increasing

$\Leftrightarrow f''(x) > 0$



local minima



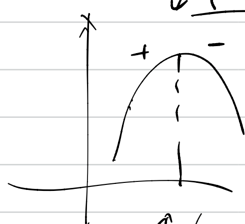
local minima

concave downward

behavior of $f'(x)$

$\Leftrightarrow f'(x)$ is decreasing

$f''(x) < 0$

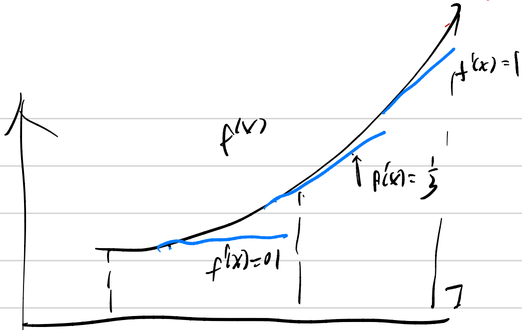


local max

What does f'' say about f ?

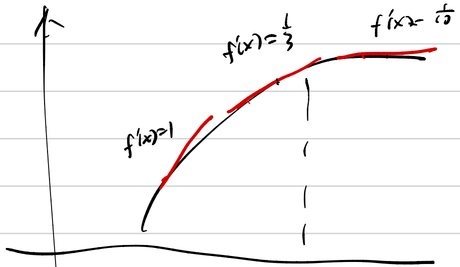
Concavity test:

a) if $f''(x) > 0$ on interval I then f is concave upward on I

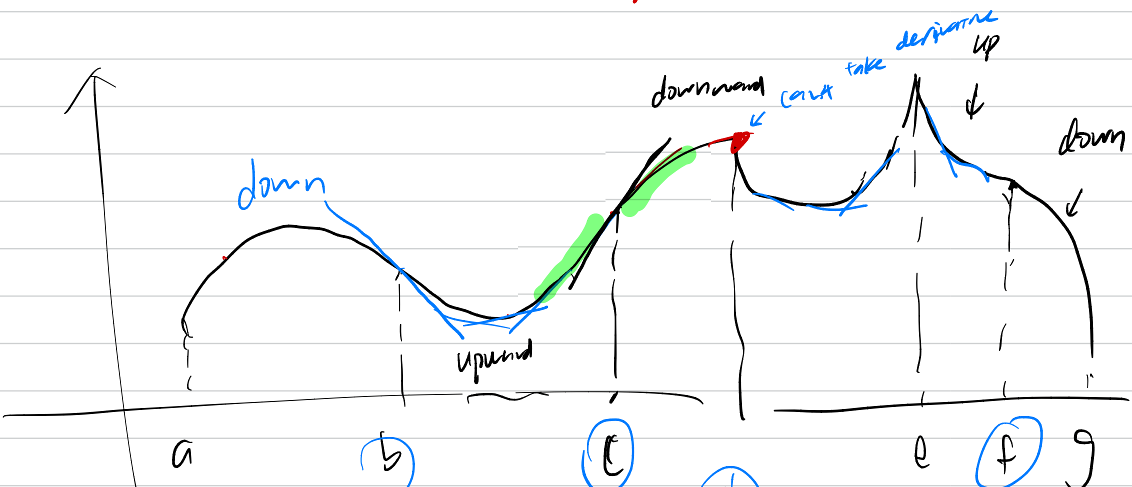


a) concave upward: curve above tangent line

b) if $f''(x) < 0$ on interval I. then graph of f is concave downward on I



b) concave downward: curve below tangent line



inflection point

if tangent line exist, curve crosses its tangent line at inflection point.

Define: A point P on a curve $y=f(x)$ is called inflection pt if f is cts there and the curve change from concave upward to concave downward or from concave downward to concave upward at P.

Example 5:

sketch graph of function by given conditions

① $f'(x)>0$ on $(-\infty, 1)$. $f'(x)<0$ on $(1,\infty)$

f(x) increasing   f(x) decreasing

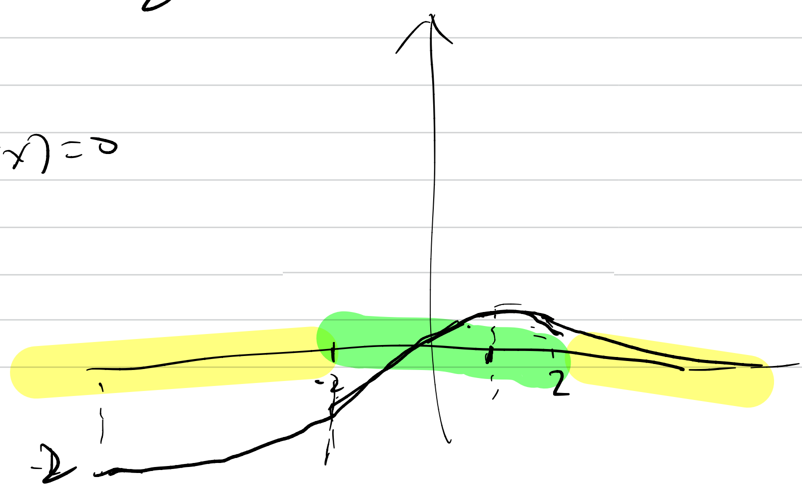
② $f''(x)>0$ on $(-\infty,-2)$ & $(2,+\infty)$

$f''(x)<0$ on $(-2,2)$

concave down   concave up

③ $\lim\limits_{x\to-\infty} f(x)=-2$

$\lim\limits_{x\to\infty} f(x)=0$



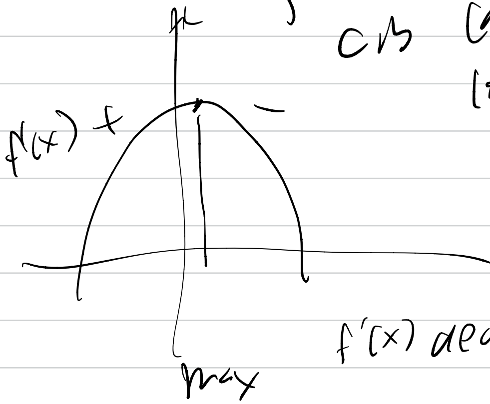
Second derivative test

Suppose f'' is cts near c

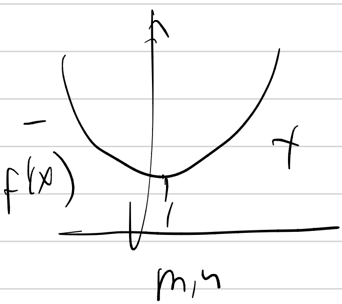
a) $f'(c)=0$ and $f''(c)>0$ f has local min

b) $f'(c)=0$ and $f''(c)<0$ f has local max

c is candidate for local min/max



$f'(x)$ decreasing $\iff$ $f''(x)<0$



$f'(x)$ increasing $\iff$ $f''(x)<0$

when $f''(c)=0$ inconclusive!

when $f''(c)$ doesn't exist, test fail

apply first derivative test!!

$x \to -\infty \quad f(x) \to +\infty$

$x \to +\infty \quad f(x) \to +\infty$

$f(x) = x^3(x-3) \to +\infty$

$3 > x > 0 \sim f'(x) < 0$

$\boxed{f(x) = x^4 - 4x^3}$

$f'(x) = 4x^2(x-3)$

$f'(x) = 4x^3 - 12x^2$

$x < 0 \sim f'(x) < 0$

sign of $f'(x)$ doesn't change, 0 is not local max/min

$\boxed{f'(x) = 0 = 4x^3 - 12x^2} = x^2(4x - 12)$

$= 4x^2(x-3)$

$x_1 = 0 \qquad x_2 = 3$

$f''(x) = 12x^2 - 24x$ Defined everywhere, CTS everywhere

$= 12x(x-2)$

$x = 0$ inflection point

when $x < 0$ $f''(x) > 0$

$2 > x > 0$ $f''(x) < 0$

$x > 2$ $f''(x) > 0$

$x = 2$

apply 2nd derivative test

$f''(0) = 12 \times 0 - 24 \times 0 = 0$ inconclusive

apply first derivative test

$f''(3) = 12 \times 9 - 24 \times 3$

$= 108 - 72 = 36 > 0$

3 is a local min

$x \to +\infty \quad f(x) \to +\infty$

$x \to -\infty \quad f(x) \to -\infty$

$f(x) = x^3 - 3x^2 + 5$

① Find interval when f is increasing / decreasing

$f(x)$ increasing on $(-\infty, 0)$ $(2, +\infty)$

decreasing on $(0, 2)$

② Find ... where f concave up or down

$x > 1$ up $\quad x < 1$ down

③ Find inflection point of f

1 is inflection point

④ Sketch graph

$f'(x) = 3x^2 - 6x = 3x(x-2)$ increasing decreasing increasing

$x = 0 \quad x = 2$

$f'(x) > 0 \quad f'(x) < 0 \quad f'(x) > 0$

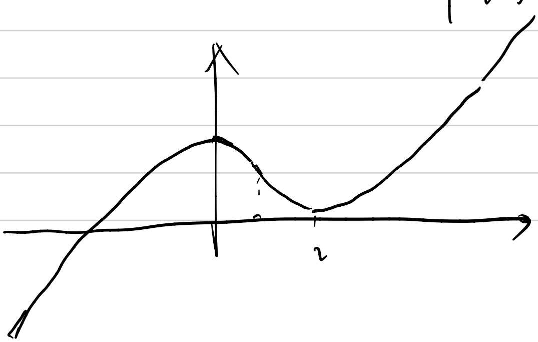
0 $\qquad$ 2

$f''(x) = 6x - 6$

$x = 1$

down $\quad$ inflection $\quad$ up

$f''(x) < 0$ max $\quad$ 1 $\quad$ min $f''(x) > 0$



$x \to +\infty$ $\quad f(x) \to +\infty$

$x \to -\infty$ $\quad f(x) \to -\infty$

$$f(x) = x^3 - 6x^2 - 135x$$

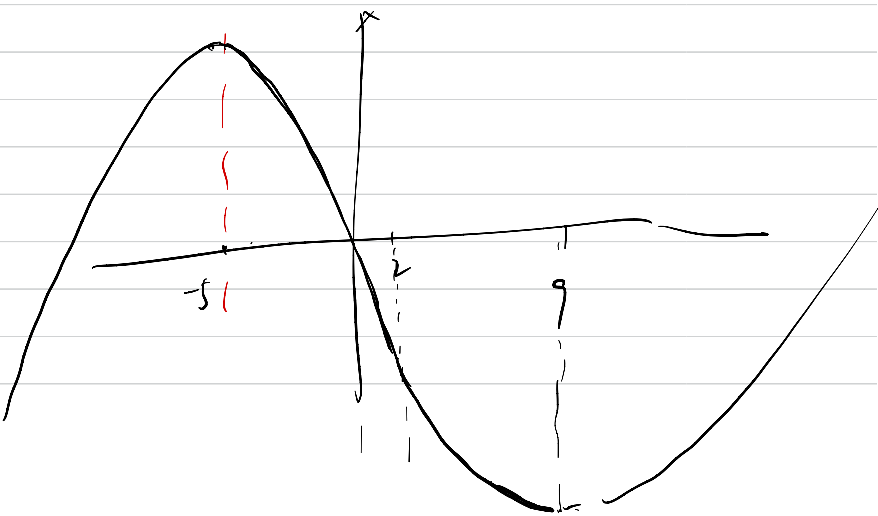
$$-45 = -9 \times 5$$

$$f'(x) = 3x^2 - 12x - 135$$
$$= 3(x^2 - 4x - 45)$$
$$= 3(x-9)(x+5)$$

$f(x)$	increasing	max	decreasing	min	increasing
$f'(x)$	$+$	-5	$-$	9	$+$

$$f''(x) = 6x - 12$$

	down	inflection point	up
$f''(x)$	$-$	2	$+$



$$f(x) = x^{2/3}(6-x)^{\frac{1}{3}}$$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}} - x^{\frac{2}{3}}\frac{1}{3}(6-x)^{-\frac{2}{3}}$$

$$= \frac{2(6-x)^{\frac{1}{3}}}{3x^{\frac{1}{3}}} - \frac{x^{\frac{2}{3}}}{3(6-x)^{\frac{2}{3}}}$$

$$= \frac{2(6-x)^{\frac{1}{3}}(6-x)^{\frac{2}{3}} - x^{\frac{2}{3}}x^{\frac{1}{3}}}{3x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}} = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$$

$$= \frac{2(6-x)-x}{3x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}} = \frac{12-3x}{3x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$$

when $x=4$ $f'(4)=0$

$x=0$, $x=6$. $f'(x)$ DNE

$$f''(x) = \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$$

$f(x)$ decreasing increasing decreasing decreasing

0 4 6

$f'(x)$ − min + max − 6 −

down down up

0 − 6 +

$f''(x)$ −

inflection point

$x \to -\infty \quad f(x) \to +\infty$

$x \to +\infty \quad f(x) \to -\infty$

$$f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$$

$$f'(x) = \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}}$$

$$f''(x) = \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$$

f(x): decreasing | increasing | decreasing | decreasing

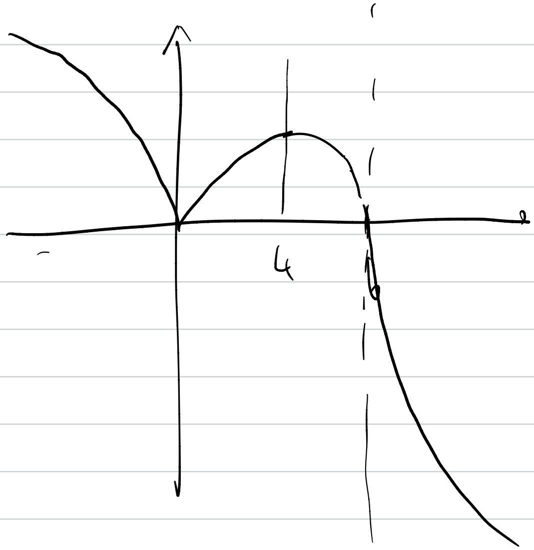
f'(x): − (0) + (4) − (6) −

min at 0, max at 4

down | down | up

f''(x): − (0) − (6) +

inflection point



$f(x) = e^{\frac{1}{x}}$ Domain $: \mathbb{R} - \{0\}$

$\lim_{x \to 0^+} e^{\frac{1}{x}} = e^{\lim_{x \to 0^+} \frac{1}{x}} = e^{+\infty} = +\infty$

$\lim_{x \to 0^-} e^{\frac{1}{x}} = e^{\lim_{x \to 0^-} \frac{1}{x}} = e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$\lim_{x \to \infty} e^{\frac{1}{x}} = e^{\lim_{x \to \infty} \frac{1}{x}} = e^{0} = 1$

$\lim_{x \to -\infty} e^{\frac{1}{x}} = 1$

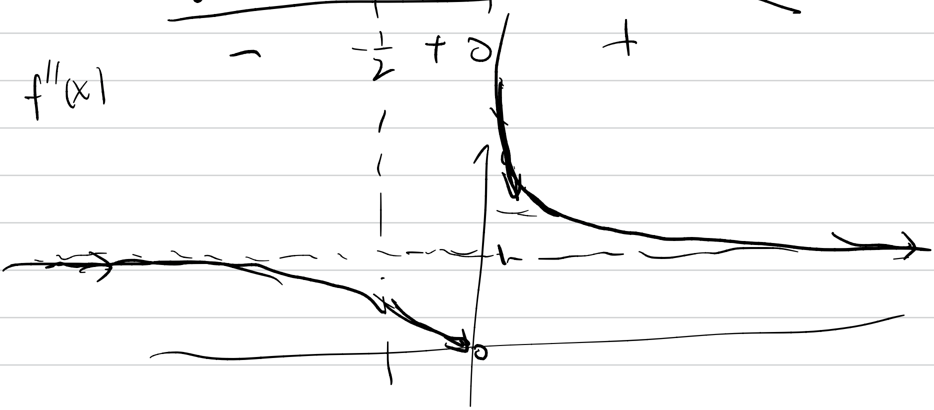
$f'(x) = e^{(\frac{1}{x})} \cdot \frac{-1}{x^2} = \frac{-e^{\frac{1}{x}}}{x^2}$ $f''(x) = \frac{e^{\frac{1}{x}}(2x+1)}{x^4}$

$f(x)$ decreasing decreasing

$f'(x)$ $-$ 0 $-$

down up up

$f''(x)$ $-$ $-\frac{1}{2}$ $+$ 0 $+$



L'Hopital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim(g(x))}$$

exist

denominator $\neq 0$

When denominator $= 0$, can't apply for quotient law.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

learned before (by geometric method)

not able to use quotient law $\frac{0}{0}$

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$ both $f(x) \to 0$ $g(x) \to 0$ as $x \to a$

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$$

called an ~~indeterminate~~ form of type $\frac{0}{0}$

both $f(x) \to \infty$ $(-\infty)$ $g(x) \to \infty$ $(-\infty)$ as $x \to a$

of type $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\left(\frac{x^2-1}{2x^2+1}\right) = \lim_{x\to\infty}\left(\frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}}\right) = \frac{1}{2}$$

Suppose f and g are differentiable

$\boxed{g'(x) \neq 0}$ on an open interval $I \ni a$.

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ $\left(\frac{0}{0}\right)$

or

$\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$ $\left(\frac{\infty}{\infty}\right)$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{2x}{4x} = \lim_{x \to \infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty} \frac{\ln x}{x-1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2}$$

L'Hopital's rule

$$= +\infty$$

$$\left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2(\sec x)\sec x \tan x}{6x}$$

$\tan x \to 0$ as $x \to 0$

$\sec^2 x = \frac{1}{\cos^2 x} \to 1$ as $x \to 0$

$x \to 0$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \lim \frac{\tan x}{x}$$

$\to 1$ $\quad$ $\frac{\sin x}{x \cos x}$ $\to 1$ $\quad$ $\to 1$

$$= \frac{1}{3}$$