$$\frac{y}{x} \cdot \left(\frac{-x}{y}\right) = -1$$

06/12

Chain Rule, Implicit differentiation

$$\boxed{x^2 + y^2 = 25}$$

$$\frac{d(x^2+y^2)}{dx} = \frac{d\,25}{dx}$$

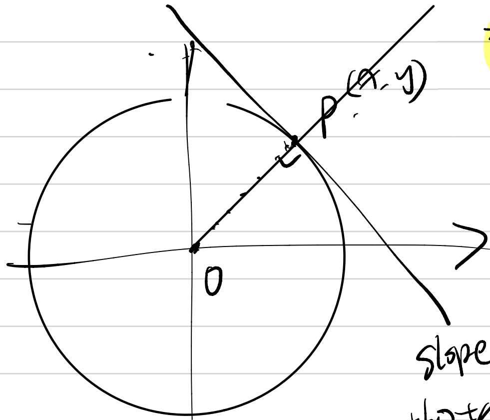
$$\boxed{2x + 2y \cdot y' = 0}$$

$$2y \cdot y' = -2x$$

$$y' = \frac{-x}{y}$$

$\log_b x$

slope of OP $= \frac{y-0}{x-0} = \frac{y}{x}$



slope of this tangent line $\left(\frac{-x}{y}\right)$

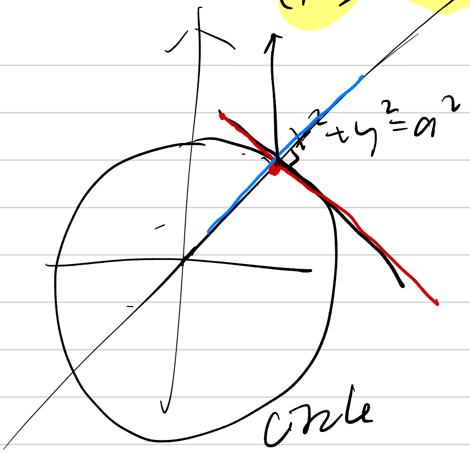
$x^2 + y^2 = r^2$ general equation for a circle

$$\frac{d(x^2+y^2)}{dx} = \frac{d(r^2)}{dx}$$

$$2x + 2y \cdot y' = 0$$

$$\boxed{y' = \frac{-x}{y}}$$

$(x, y) = \left(x, -\frac{ax}{b}\right)$ $y = kx$



two curve are orthogonal if their tangent lines are perpendicular at each point of intersection

① $x^2+y^2=r^2$

↓

implicit differentiation

$\boxed{y' = -\frac{x}{y}}$

derivative of circle

straight line passing through origin

$ax+by=0$

↓

$\frac{d(ax+by)}{dx} = 0$

$a + by' = 0$

$y' = \left(\frac{-a}{b}\right)$

$$\begin{cases} x^2+y^2=r^2 \\ ax+by=0 \end{cases}$$

$\boxed{y = -\frac{ax}{b}}$

since (x, y) is located on straight line

$y = \frac{-ax}{b} \Rightarrow y' = \frac{-x}{-\frac{ax}{b}} = \frac{b}{a}$

slope of tangent line of circle at intersection point

$\boxed{x^2+y^2=ax}$

$x^2-ax+y^2=0$

$x^2-ax+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$

$(x-\frac{a}{2})^2-\frac{a^2}{4}+y^2=0$

$(x-\frac{a}{2})^2+y^2=\frac{a^2}{4}$

$\boxed{x^2+y^2=by}$

$x^2+y^2-by=0$

perfect square

$x^2+y^2-by+\frac{b^2}{4}=\frac{b^2}{4}$

$x^2+(y-\frac{b}{2})^2=(\frac{b}{2})^2$

for intersection point

$\boxed{ax=by}$

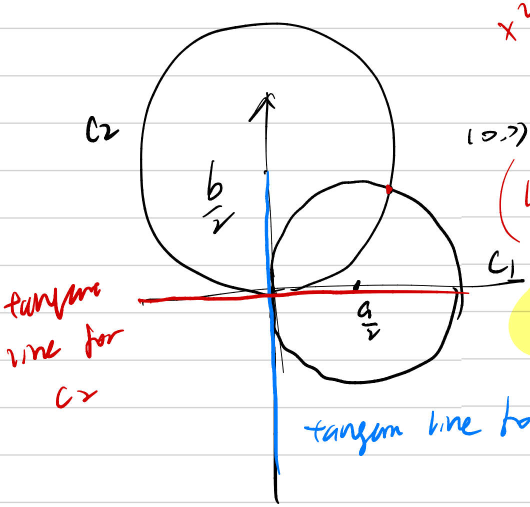
$x_1=0$

$x^2+(\frac{ax}{b})^2=ax$

$(1+\frac{a^2}{b^2})x^2=ax$

$(1+\frac{a^2}{b^2})x=a$

$x=\frac{a}{1+\frac{a^2}{b^2}}$



$$\begin{cases} x^2+y^2=ax \\ x^2+y^2=by \end{cases}$$

$x^2+y^2=by$

$(x^2+y^2)'=(by)'$

$2x+2yy'=by'$

$(2y-b)y'=-2x$

$y'=\frac{-2x}{2y-b}$

$x^2+y^2=ax$

Implicit differentiation method

$(x^2+y^2)'=(ax)'$

$2x+2yy'=a$

$\boxed{y'=\frac{a-2x}{2y}}$

$y=\frac{ax}{b}$

$y'=\frac{a-2x}{2(\frac{ax}{b})}=\frac{2b(a-2x)}{2ax}$

$y'=\frac{-2x}{\frac{2ax}{b}-b}=\frac{-2bx}{2ax-b^2}$

$x^2 - xy - y^2 = 1$ slope of tangent line (2,1)

① take differentiation on both side

$(x^2 - xy - y^2)' = (1)'$

$\boxed{2x - (xy' + y \cdot 1) - 2yy' = 0}$ ②

← compute derivative

$2x - xy' - y - 2yy' = 0$ ③ expand every thing

④ find out item contain y'

$2x - y = xy' + 2yy'$

$2x - y = y'(x + 2y)$

$\boxed{y' = \frac{2x - y}{x + 2y}}$

⑤ move every item contain to the one side

left every thing without y' to the other side

⑥ take out common factor y'

and divide coefficient of y'

$y' = \frac{2 \times 2 - 1}{2 + 2 \times 1} = \frac{3}{4}$

$\log_b x$

Inverse function of b^x

Since graph of $\log_b x$ is symmetric with graph of b^x according to $y=x$

therefore. $\log_b x$ is differentiable

$$\boxed{\frac{d \log_b x}{dx} = \frac{1}{x \ln b}}$$

Special case when $b=e$

$\ln b = \ln e = 1$

$\log\left(\frac{a}{b}\right) = \log a - \log b$

$$\boxed{\frac{d(\ln x)}{dx} = \frac{1}{x}}$$

$$\frac{d}{dx}\left(\ln\left(\frac{x+1}{\sqrt{x-2}}\right)\right)$$

Lots of work here.

$$= \frac{1}{\left(\frac{x+1}{\sqrt{x-2}}\right)} \cdot \frac{d}{dx}\left(\frac{x+1}{\sqrt{x-2}}\right)$$

$$\frac{1}{x+1} - \frac{1}{2(x-2)}$$

$//$

$$= \frac{d}{dx}\left(\ln(x+1) - \ln(\sqrt{x-2})\right)$$

$$= \frac{d}{dx}(\ln(x+1)) - \frac{d}{dx}(\ln(\sqrt{x-2})) = \frac{1}{x+1} - \frac{1}{\sqrt{x-2}} \cdot \frac{1}{2\sqrt{x-2}}$$

$\ln(x)$ has domain $(0, +\infty)$

$f(x) = \ln|x|$

domain $(-\infty, \infty) / \{0\}$

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}}$$

$$f(x) = \begin{cases} \ln x & \text{when } x > 0 \\ \ln(-x) & \text{when } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{x} & \text{when } x > 0 \\ \frac{1}{(-x)} \cdot (-1) & \text{when } x < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{x} & \text{when } x > 0 \\ \frac{1}{x} & \text{when } x < 0 \end{cases} = \frac{1}{x}$$

$f'(x) = \frac{1}{x}$ for every $x \neq 0$

domain $(-\infty, \infty) - \{0\}$

$$y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$$

y' ? quotient rule ?

Lots of work(!)

Take $\ln$ at both side

$$\ln y = \ln\left(\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$$= \ln\left(x^{\frac{3}{4}}\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right)$$

$$= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

take differenentization on both side

$$\frac{1}{y}\cdot y' = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}\cdot(2x) - 5\cdot\frac{1}{3x+2}\cdot 3$$

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

derivative function of y'

if we want explicity

substitute $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

back to answer

$$y' = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Log diff (ln)

① Take natural log of both side $y = f(x)$
use law of log to expand the expression

② Differentiate implicitly with respect to x

③ Solve y' and replace y by $f(x)$

$$y = \frac{x^{\frac{5}{4}}\sqrt{x^2+1}}{(3x^2+2)^5}$$ already assume $x \geq 0$, $y \geq 0$

but $y = \frac{x^3}{3x+4}$ How can we use method of log. diff?

Then we need add extra step

$y = f(x) \Rightarrow |y| = |f(x)|$

We can repeat our method

$\ln|y| = \ln|f(x)|$ $\frac{y'}{y} = \ln|f(x)|$

use this to solve general version for power rule

$y = x^n \qquad \frac{dy}{dx} = nx^{n-1}$

$n \in \mathbb{R}$.

have proof it for n integer

$y = x^n$

n can be any number

$>0, <0, =0$

here y can be negative

in order to take $\ln$, first take absolute value

$|y| = |x^n| \Rightarrow \ln|y| = \ln|x^n|$

$= \ln|x|^n$

$= n\ln|x|$

$|a^n| = |a|^n$

law of $|\cdot|$

$$\boxed{\ln|y| = n \cdot \ln|x|} \quad \boxed{y = x^n}$$

Rember fact

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

Implicit differentiation method.

$$(\ln|y|)' = (n\ln|x|)'$$

$$\frac{1}{y}\cdot y' = n\left(\frac{1}{x}\right)$$

$$y' = n \cdot \frac{y}{x} = n \cdot \frac{x^n}{x}$$

$$y' = n \cdot x^{n-1}$$

Method ①

$y = x^{\sqrt{x}} = \left(e^{\ln x}\right)^{\sqrt{x}} \Rightarrow y = e^{\ln x \cdot \sqrt{x}}$

$e^{\ln x} = x$

$e^{(\cdot)}$ familiar

chain rule $\Rightarrow$ $\left\{ \dfrac{e^{(\cdot)}}{(\ln x \cdot \sqrt{x})} \right.$ product rule

$y = e^{\ln x \cdot \sqrt{x}}$

$$\frac{dy}{dx} = e^{\ln x \cdot \sqrt{x}} \cdot \frac{d}{dx}\left(\ln x \cdot \sqrt{x}\right)$$

$= x$

$$= x^{\sqrt{x}}\left(\ln x \cdot \frac{1}{2\sqrt{x}} + \sqrt{x} \cdot \frac{1}{x}\right)$$

$$= x^{\sqrt{x}}\left(\frac{\ln x}{2} \cdot \frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{x}\right)$$

$$= x^{\sqrt{x}}\left(\frac{\ln x}{2} + 1\right)\left(\frac{1}{\sqrt{x}}\right)$$

Method②

$x \geq 0$ to make sure $\sqrt{x}$ make sense

domain $x \geq 0$ $y \geq 0$

$$y = x^{\sqrt{x}}$$

$$\ln(y) = \ln(x^{\sqrt{x}})$$

$$\ln(y) = \sqrt{x}\ln(x)$$

differentiate both side

$$\frac{y'}{y} = \frac{d}{dx}\left(\sqrt{x}\cdot\ln(x)\right)$$

$$= \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}$$

$$y = x^{\sqrt{x}}$$

$$y' = y\left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right)$$

e is point satisfy $\frac{d}{dx}(e^x)\Big|_{x=0} = 1$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\quad \boxed{f(x) = \ln x}$

$\frac{d}{dx}(\ln x)\Big|_{1} = \frac{1}{1} = 1$ $\quad \boxed{f'(1) = 1}$

$$1 = \boxed{\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}} = \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h}$$

definition of derivative

$$= \lim_{h\to 0} \frac{\ln(1+h)}{h}$$

$$= \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{1}{x} \cdot \ln(1+x) = 1$$

$b\ln(a) = \ln(a^b)$

law of log

$$\frac{\lim_{x\to 0} \ln\left((1+x)^{\frac{1}{x}}\right)}{} = 1$$

$\lim_{x\to a} g(x) = b$

$\lim_{x\to a} f\circ g(x) = f(b)$

$= f\left(\lim_{x\to a} g(x)\right)$

Theorem based on continuity

$g(x)$

$$e^{\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}}} = e^1$$

$f(x)$

$$LHS = \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = RHS = e$$

$\Downarrow$

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

$n = \frac{1}{x}$ $\quad x\to 0 \Downarrow n\to\infty$

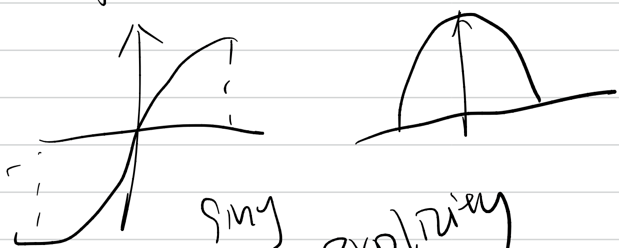
$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$\boxed{y = \arcsin x \text{ means } \sin y = x \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad -1 \le x \le 1}$

How can we find $\frac{d \arcsin x}{dx}$? $= \frac{dy}{dx} = y'$

$\sin y = x$ using Implicit differentiation

$\frac{d(\sin y)}{dx} = \frac{dx}{dx}$



$\sin y$

$\cos y \cdot y' = 1$

we need it explicitly

$\Downarrow$

$y' = \frac{1}{\cos y}$

$\sin y = x$

$\boxed{-\frac{\pi}{2} \le y \le \frac{\pi}{2}}$

$\cos^2 y + \sin^2 y = 1$

$\cos y = \sqrt{1 - \sin^2 y}$

$= \sqrt{1 - x^2}$

$\cos y \ge 0$ for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y' = \frac{1}{\sqrt{1 - x^2}}$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$y = \arctan x$ means $\tan y = x$

$$\frac{d(\tan y)}{dx} = \frac{dx}{dx} \qquad y' = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$$

$$\sec^2 y \cdot y' = 1$$

$$\sec^2 y - \tan^2 y = \frac{1}{\cos^2 y} - \frac{\sin^2 y}{\cos^2 y}$$

$$= 1$$

$$\sec^2 y = 1 + \tan^2 y$$

$$= 1 + x^2$$

$$\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$$

① $y = \frac{1}{\arcsin x}$ $\qquad y'$

chain rule or quotient rule

$$y' = -(\arcsin x)^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{-1}{\arcsin^2 x \sqrt{1-x^2}}$$

---

② $f(x) = x \arctan \sqrt{x}$

product rule

$$f'(x) = x \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} + \arctan\sqrt{x}$$

$$= \frac{x}{2(1+x)\sqrt{x}} + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1+x)} + \arctan\sqrt{x}$$

$$g(x) = \operatorname{arcsec}(x^2)$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{dg(x)}{dx} = \frac{1}{(x^2)\sqrt{(x^2)^2 - 1}} \cdot (2x)$$

$$= \frac{2x}{x^2\sqrt{x^4 - 1}}$$

$$= \frac{2}{x\sqrt{x^4 - 1}}$$

$$y = e^{\arcsin(x^2)} \qquad y'$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = e^{(\arcsin(x^2))} \quad \frac{d}{dx}\left(\arcsin(x^2)\right)$$

$$= e^{\arcsin(x^2)} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$= e^{\arcsin(x^2)} \frac{2x}{\sqrt{1-x^4}}$$

$$y = \arctan\left(x - \sqrt{1+x^2}\right)$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$y' = \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2} \cdot \frac{d}{dx}\left(x - \sqrt{1+x^2}\right)$$

$$= \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2} \cdot \left(1 - \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= \frac{1}{1+\left(x-\sqrt{1+x^2}\right)^2} \cdot \left(1 - \frac{x}{\sqrt{1+x^2}}\right)$$

$y = x^{\sin x}$ log of differentiation

y'

$\ln|y| = \ln(|x^{\sin x}|)$

no need add |·| here

$\ln|y| = \ln(|x|^{\sin x})$

$\ln|y| = \underline{\sin x \ln(|x|)}$

product rule

↓ take derivative

$$\frac{y'}{y} = \sin x \cdot \frac{1}{x} + \cos x \cdot \ln(|x|)$$

$$y' = y\left(\frac{\sin x}{x} + \cos x\, \underline{\ln|x|}\right)$$

constant c. $c' = 0$. $x' = 1$

$(f+g)' = f' + g'$

$(x^n)' = n \cdot x^{n-1}$

$(f \cdot g)' = fg' + g \cdot f'$

$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f g'}{g^2}$

$\frac{d}{dx}(cf) = c \cdot \frac{d}{dx} f$

$(f \circ g)' = f'(g(x)) \cdot g'(x)$

using chain rule to prove

$\frac{d}{dx}(b^x) = \ln b \cdot (b^x)$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \tan x \sec x$

Implicit differentiation

We cannot explicitly write the function $y = f(x)$

$\Downarrow$

$\frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} \log_b x = \frac{1}{\ln b \cdot x}$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

---

① $e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

② $\lim_{x\to 0}\frac{\sin x}{x} = 1$ $\qquad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$

---

property of ln:

property of exponential function

$\ln(ab) = \ln(a) + \ln(b)$

$\ln\left(\frac{x+2}{x+3}\right) = \ln(x+2) + \ln(x+3)$

↑ easier to differentiate