6.10 Monday

Product rule  Quotient rule  derivative of Trig

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$  $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

$(f \cdot g)' = \underline{f \cdot g' + g \cdot f'}$  product rule

$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$  Quotient rule

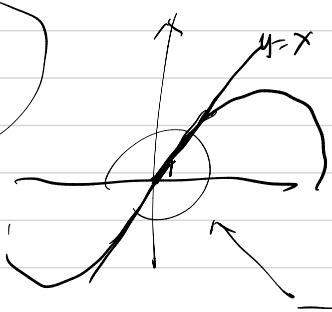
$(\sin x)' > \cos x$

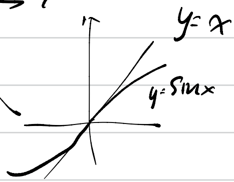
$(\cos x)' = -\sin x$

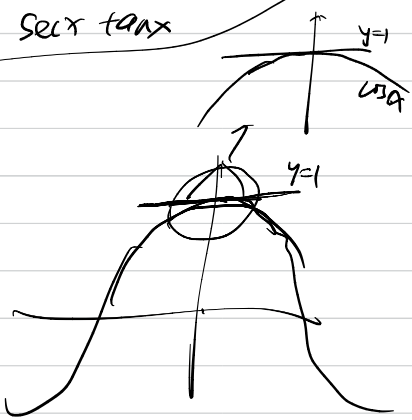
$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{1}{\cos^2 x} = \sec^2 x$

$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \sec x \tan x$

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$




$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

Quotient rule

$$y = \frac{t\sin t}{1+t} \qquad y' = \frac{(1+t)\frac{d}{dt}(t\sin t) - t\sin t\,\frac{d}{dt}(1+t)}{(1+t)^2}$$

$=1$

$$\frac{d}{dt}(t\sin t) = t\cos t + \sin t \cdot 1$$

$$= t\cos t + \sin t$$

$$\frac{(1+t)(t\cos t + \sin t) - t\sin t}{(1+t)^2}$$

$f(t) = \cos^2 t$ (① chain rule)

$$= \frac{\cos(2t)+1}{2}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1-\cos^2\theta) = \cos^2\theta - 1 + \cos^2\theta$$

$$= 2\cos^2\theta - 1$$

$$\cos(2\theta) + 1 = 2\cos^2\theta$$

$$\cos^2\theta = \frac{\cos(2\theta)+1}{2}$$

$$f'(t) = \left(\frac{\cos(2t)}{2}\right)' + \left(\frac{1}{2}\right)'$$

$$= \frac{1}{2}\left(\cos(2t)\right)'$$

$$2\lim_{h\to 0}\left(\frac{\cos(2(t+h)) - \cos(2t)}{2h}\right)$$

$$= 2\lim_{2h\to 0}\frac{\cos(2t+2h) - \cos(2t)}{2h}$$

$$= 2\ -\sin(2t)$$

$$\cos t' = -\sin t$$

$\cancel{\cos(2t)' = -\sin(2t)}$ ?

$$= -\sin(2t)\cdot 2$$

$$= \frac{1}{2}\cdot 2(-\sin(2t)) = -\sin(2t)$$

# chain rule

$(f \circ g(x))' = ?$

$g+f \quad f \cdot g \quad \frac{f}{g}$

$$(f\circ g(x))' = \lim_{\Delta x \to 0} \frac{\Delta (f \circ g(x))}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x}$$

$\Delta g = g(x+\Delta x) - g(x)$

$\Delta f = f(g+\Delta g) - f(g)$

$$= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \quad \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x}$$

g is its function

$\Delta x \to 0$

$\Downarrow$

$\Delta g \to 0$

$$= \lim_{\Delta g \to 0} \frac{\Delta f}{\Delta g} \cdot g'(x)$$

$$= f'(g) \cdot g'(x)$$

$$= f'(g(x))\, g'(x)$$

If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$

Then: $F = f \circ g$ $\quad F(x) = f(g(x))$ is differentiable at $x$.

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$y = f(u)$ & $u = g(x)$ both differentiable

easy to remember

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\frac{dy}{du} = f'(u(x)) = f'(g(x))$

$\frac{du}{dx} = g'(x)$

$y = f(u) \quad u = g(x) \quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$h(t) = \cos^2 t$

$y = f \circ g(t)$

$= [\cos t]^2 = h(t)$

$u = g(t) = \cos t$

$y = f(u) = u^2$

chain rule

$\dfrac{dy}{du} = \dfrac{du^2}{du} = 2u$

$\dfrac{du}{dt} = \dfrac{d\cos t}{dt} = -\sin t$

$\dfrac{d\cos^2 t}{dt} = \dfrac{dy}{dt} = \dfrac{dy}{du} \cdot \dfrac{du}{dt}$

$= 2\cos t(-\sin t)$

$= -2\cos t \sin t$

$= -\sin 2t$

sqrt

$F(x) = \sqrt{x^2+1}$

polynomial

$F' = f'(g(x)) \cdot g'(x)$

① first definition

$f(x) = \sqrt{x}$

$F = f \circ g$

$= \dfrac{1}{2\sqrt{g(x)}} \cdot 2x$

$g(x) = x^2+1$

$= \dfrac{1}{2\sqrt{x^2+1}} \cdot 2x$

$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

$= \dfrac{2x}{2\sqrt{x^2+1}} = \dfrac{x}{\sqrt{x^2+1}}$

$g'(x) = 2x$

② second notation

definition

$\boxed{y = \sqrt{u}} \quad \boxed{u = x^2+1}$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \left(\frac{1}{2} u^{\frac{1}{2}-1}\right) \times (2x)$

$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \dfrac{x}{\sqrt{x^2+1}}$

$f(x) = \sqrt{1+x^2}$

$f'(x)$ ⟶ $f(u)$

① Identify $u$. $u = 1+x^2$

② Write $f(u) = \sqrt{u}$

③ thinking about chain rule $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$   $f'(x)$

④ Compute $\frac{df}{du} = \frac{1}{2\sqrt{u}}$ and $\frac{du}{dx} = 2x$

⑤ Substitute using chain rule $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{2x}{2\sqrt{u}}$

function with variable $u$

⑥ Replace $u$ by $x$.

$$\frac{df}{dx} = \frac{2x}{2\sqrt{1+x^2}}$$

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$y = \sin(x^2)$   $y'$

$u = x^2$   $y = f(u) = \sin(u)$

$\frac{dy}{du} = \cos u$, $\frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos u \cdot 2x$$

$$= \cos(x^2) \cdot 2x$$

sin to be outer function

derivative of outer function

$y = \sin(x^2)$    $y' = \cos(x^2) \cdot 2x$

$x^2$ to be inner function

evaluate at inner function

evaluate at inner

derivative of inner function

$(f \circ g(x))' = f'(g(x))\, g'(x)$

outer function

inner function

derivative of outer

derivative of inner function

$y = (\sin x)^2$

$y' = 2(\sin x)^{2-1} \cos x$

$= 2 \sin x \cos x$

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1} f'(x)$$

$$y = (x^3-1)^{100} \qquad y' = \underline{100}(x^3-1)^{99}(\underline{3}x^2)$$

$$= 300(x^3-1)^{99}x^2$$

$$f(x) = \frac{1}{\sqrt[3]{x^2+x+1}} \qquad f'(x) = -\frac{1}{3}(x^2+x+1)^{-\frac{1}{3}-1}(2x+1)$$

$$= (x^2+x+1)^{-\frac{1}{3}} \qquad = -\frac{(2x+1)}{3(x^2+x+1)^{\frac{4}{3}}}$$

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9$$

$$= \left(\frac{t-2}{2t+1}\right)^9 = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{(2t+1)\cdot 1-(t-2)\cdot 2}{(2t+1)^2}$$

$$2t+1-(2t-4) = 2t+1-2t+4 = 5$$

$$= \frac{9(t-2)^8 \times 5}{(2t+1)^8\,(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}$$

$$y = (2x+1)^5 (x^3 - x + 1)^4$$

$$y' = (2x+1)^5 \frac{d}{dx}\left((x^3 - x + 1)^4\right) + \frac{d}{dx}\left((2x+1)^5\right)(x^3 - x + 1)^4$$

$$= (2x+1)^5 \frac{d}{dx}\left((x^3 - x + 1)^4\right) + \frac{d}{dx}\left((2x+1)^5\right)(x^3 - x + 1)^4$$

$$= (2x+1)^5 \, 4(x^3 - x + 1)^3 (3x^2 - 1) + 5(2x+1)^4 \cdot 2 \, (x^3 - x + 1)$$

$f(x) = e^x \qquad f'(x) = e^x$

$y = e^{\sin x} \qquad y' = e^{(\sin x)} \cos x$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dt}$$

$$f(x) = \sin(\cos(\tan(x)))$$

chain

$$= \cos(\cos(\tan(x))) \frac{d}{dt}\left(\cos(\tan(x))\right)$$

chain rule

$$= \cos(\cos(\tan(x))) \cdot (-\sin(\tan(x))) \sec^2 x$$

$$= -\cos(\cos(\tan x)) \sin(\tan(x)) \cdot \sec^2 x$$

$$y = e^{(\sec 3\theta)}$$

$$= e^{(\sec 3\theta)} \times \frac{d}{d\theta}\left( \sec(3\theta) \right)$$

$$= e^{\sec 3\theta} \cdot \sec(3\theta)\tan(3\theta) \cdot 3$$

$$= 3e^{\sec 3\theta} \cdot \sec(3\theta)\tan(3\theta)$$

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$f(x) = e^x$ $\quad f'(x) = e^x$ $\quad b^x = \left(e^{\ln b}\right)^x$

$$\boxed{e^{\ln b} = b} = e^{(\ln b)x}$$

constant

$$(b^x)' = \left(e^{\ln b x}\right)' = e^{(\ln b x)} \cdot \ln b$$

$$= \ln b \cdot e^{(\ln b)x}$$

$g(x) = 2^x$ $\quad b = 2$ $\quad g'(x) = (\ln 2)\, 2^x$

$h(x) = 5^{x^2} = \ln 5 \cdot 5^{(x^2)}$ $\quad 2x = 2\ln 5\, x \cdot 5^{x^2} = (\ln b)\, b^x$

$$\boxed{\frac{d}{dx} b^x = \ln b \cdot b^x}$$

$\frac{d}{dx} e^x = e^x$

when $b = e$

$\ln e = 1$

$y = (5 - x^4)^3$

$y' = 3(5-x^4)^2(-4x^3)$

$= -12(5-x^4)^2(x^3)$

$y = e^{(\sqrt{x})}$

$y' = e^{(\sqrt{x})} \frac{1}{2}\frac{1}{\sqrt{x}}$

$= \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$

$y = \sin(\cos x)$

$y' = \cos(\cos x) \cdot (-\sin x)$

$= -\cos(\cos x) \cdot \sin x$

$y = \sin x \cos(1-x^2)$

$y' = \sin x \cdot \frac{d}{dt}\left(\cos(1-x^2)\right) + \cos x \quad \cos(1-x^2)$

$= \sin x\left(-\sin(1-x^2)(-2x)\right) + \cos(1-x^2) \cos x$

$= \sin x \sin(1-x^2) \cdot 2x + \cos(1-x^2)\cos x$

$y = \sin(\sin x)$

find tangent line at $(\pi, 0)$

$y' = \cos(\sin(x)) \cos(x)$

$\uparrow$

$x = \pi$

$y'(\pi) = \cos(\sin(\pi)) \cdot \cos(\pi)$

$= \cos(0) \quad \cos(\pi) = 1 \cdot (-1) = -1$

slope $= -1$, $(\pi, 0)$

$y - 0 = (-1)(x - \pi)$

$y = -(x - \pi)$

$$y = b^x$$

$$y' = \ln b \cdot b^x$$

$$y = 2^x \quad (0,1)$$

$$= e^{(\ln 2 \cdot x)} = e^{(\ln 2 \cdot x)}$$

$$\ln 2 = 2^x \cdot \ln 2$$

$$\boxed{y' = \ln 2 \cdot \times 2^x}$$

when $x = 0$

$$y'(0) = \ln 2 \cdot 2^0 = \ln 2$$

slope $\ln 2$

$(0,1)$

$$y - 1 = \ln 2 (x - 0)$$

$$y - 1 = (\ln 2) x$$

$$y = \ln 2 \, x + 1$$