MTH 141

Midterm 2

June 12, 2023

Name: SOLUTIONS

UR ID: ______________________

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Instructions:

- The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________

QUESTION	VALUE	SCORE
1	14	
2	16	
3	12	
4	16	
5	18	
6	12	
7	12	
TOTAL	100	

Trig Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

1. (14 points)

(a) Use the **definition of the derivative** to compute $f'(2)$ where $f(x) = x^2 - x$. No points will be awarded if the derivative is computed by any other method, including l'Hopital rule.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[(2+h)^2 - (2+h)] - [2^2 - 2]}{h}$$

$$= \lim_{h \to 0} \frac{[4 + 4h + h^2 - 2 - h] - 2}{h} = \lim_{h \to 0} \frac{3h + h^2}{h} = \lim_{h \to 0} \frac{(3+h)h}{h}$$

$$= \lim_{h \to 0} 3 + h = 3$$

Answer:

3

(b) Find the equation of the tangent line to $y = f(x)$ at [redacted]. $(2, 2)$.

$$y - 2 = 3(x - 2)$$

$$y - 2 = 3x - 6$$

$$y = 3x - 4$$

Answer:

$y = 3x - 4$

2. (16 points) Find the derivatives of the following functions. Show your work and put your answer in the answer box.

(a)

$$f(x) = 4x^3 + 5\sqrt[3]{x^4} - \cos(x).$$

$$f(x) = 4x^3 + 5x^{\frac{4}{3}} - \cos x$$

$$f'(x) = 12x^2 + 5 \cdot \frac{4}{3} x^{\frac{4}{3}-1} - (-\sin x)$$

$$f'(x) = 12x^2 + \frac{20}{3} x^{\frac{1}{3}} + \sin x$$

Answer:

$$12x^2 + \frac{20}{3}x^{\frac{1}{3}} + \sin x$$

(b)

$$g(x) = \frac{\pi^e + e^x}{x^2 - 1}.$$

$$g'(x) = \frac{(\pi^e + e^x)' \cdot (x^2-1) - (\pi^e + e^x)(x^2-1)'}{(x^2-1)^2}$$

$$g'(x) = \frac{e^x(x^2-1) - (\pi^e + e^x) \cdot 2x}{(x^2-1)^2} = \frac{e^x x^2 - e^x - 2x\pi^e - 2x e^x}{(x^2-1)^2}$$

Answer:

$$\frac{e^x x^2 - e^x - 2x\pi^e - 2x e^x}{(x^2-1)^2}$$

(c)

$$h(x) = e^{x^2+\ln(\cos(x))}.$$

$$h'(x) = e^{x^2+\ln(\cos x)} \cdot \left(x^2 + \ln(\cos x)\right)'$$

$$h'(x) = e^{x^2+\ln(\cos x)} \cdot \left(2x + \frac{(-\sin x)}{\cos x}\right)$$

$$h'(x) = e^{x^2+\ln(\cos x)} \cdot (2x - \tan x)$$

Answer:

$$e^{x^2+\ln(\cos x)} \cdot (2x - \tan x)$$

(d)

$$T(x) = \arcsin\left(\frac{1}{x^2}\right).$$

Letting $u = \frac{1}{x^2} \Rightarrow \frac{du}{dx} = \frac{-2}{x^3}$

$T(u) = \arcsin u \Rightarrow \frac{dT}{du} = \frac{1}{\sqrt{1-u^2}}$

By chain rule:

$$\frac{dT}{dx} = \frac{dT}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot \frac{-2}{x^3} = \frac{-2}{\sqrt{1-\frac{1}{x^4}}\, x^3}$$

Answer:

$$\frac{-2}{\sqrt{1-\frac{1}{x^4}} \cdot x^3}$$

3. (12 points) Let $f(x) = x^2 g(x)$. Suppose that $g(1) = 2$ and $g'(1) = 3$.

(a) Find $f'(1)$.

$$f(x) = x^2 g(x)$$
$$f'(x) = 2x\, g(x) + x^2 g'(x)$$
$$f'(1) = 2g(1) + g'(1) = 2.2 + 3 = 7$$

Answer:

7

(b) Additionally, suppose that $g''(1) = 0$. Find $f''(1)$.

$$f'(x) = 2x g(x) + x^2 g'(x)$$
then $$f''(x) = 2(1.g(x) + x g'(x)) + 2x\, g'(x) + x^2 g''(x)$$
$$f''(x) = 2g(x) + 4x g'(x) + x^2 g''(x)$$
$$f''(1) = 2g(1) + 4g'(1) + g''(1)$$
$$= 2.2 + 4.3 + 0 = 16.$$

Answer:

16

4. (16 points) Use logarithmic differentiation to find the derivatives of the following functions. Show your process and justify your answer.

(a)

$$f(x) = \frac{e^x(x^3-1)}{\sqrt{x^2+2}}.$$

$$\ln(f(x)) = \ln\left(\frac{e^x(x^3-1)}{\sqrt{x^2+2}}\right) = \ln(e^x) + \ln(x^3-1) - \ln\left((x^2+2)^{\frac{1}{2}}\right)$$

$$\ln(f(x)) = x + \ln(x^3-1) - \frac{1}{2}\ln(x^2+2)$$

$$\frac{f'(x)}{f(x)} = 1 + \frac{3x^2}{x^3-1} - \frac{1}{2}\cdot\frac{2x}{x^2+2} = 1 + \frac{3x^2}{x^3-1} - \frac{x}{x^2+2}$$

$$\Rightarrow f'(x) = \left(1 + \frac{3x^2}{x^3-1} - \frac{x}{x^2+2}\right)\frac{e^x(x^3-1)}{\sqrt{x^2+2}}$$

Answer: $\left(1 + \frac{3x^2}{x^3-1} - \frac{x}{x^2+2}\right)\frac{e^x(x^3-1)}{\sqrt{x^2+2}}$

(b)

$$g(x) = (\sin(x))^{\sqrt{x}}.$$

$$\ln(g(x)) = \sqrt{x}\,\ln(\sin x)$$

$$\frac{g'(x)}{g(x)} = (\sqrt{x})'\,\ln(\sin x) + \sqrt{x}\cdot(\ln(\sin x))'$$

$$\frac{g'(x)}{g(x)} = \frac{1}{2\sqrt{x}}\ln(\sin x) + \sqrt{x}\cdot\frac{\cos x}{\sin x}$$

$$\Rightarrow g'(x) = \left(\frac{\ln(\sin x)}{2\sqrt{x}} + \sqrt{x}\cot x\right)(\sin x)^{\sqrt{x}}$$

Answer: $\left(\frac{\ln(\sin x)}{2\sqrt{x}} + \sqrt{x}\cot x\right)(\sin x)^{\sqrt{x}}$

5. (18 points) Use implicit differentiation to compute $\frac{dy}{dx}$ at $(0,0)$, where the relation between x and y is described as

$$y^3 + e^y - \sin(2x) = 5.$$

$$y^3 + e^y - \sin(2x) = 5$$

$$3y^2 \cdot y' + e^y \cdot y' - 2\cos(2x) = 0$$

$$(3y^2 + e^y)\, y' = 2\cos(2x)$$

$$\frac{dy}{dx} = y' = \frac{2\cos(2x)}{3y^2 + e^y}$$

Then, at $(0,0)$ $\quad y'(0) = \frac{2\cos(2.0)}{3.0^2 + e^0} = \frac{2\cos 0}{1}$

$$y'(0) = 2$$

Answer: 2

6. (12 points) Let

$$s(t) = \frac{t}{5} - \arctan t + 9,$$

denote the position of a particle in meters at time t in seconds.

(a) Compute the velocity and the acceleration of the particle at time t.

$$V(t) = s'(t) = \frac{1}{5} - \frac{1}{1+t^2}$$

$$a(t) = V'(t) = \left(\frac{1}{5} - \frac{1}{1+t^2}\right)' = -\left(\frac{1}{1+t^2}\right)' = \frac{2t}{(1+t^2)^2}$$

Answer:

$V(t) = \frac{1}{5} - \frac{1}{1+t^2}$, $a(t) = \frac{2t}{(1+t^2)^2}$

(b) At what time t is the particle at rest?

$$V(t) = 0$$

$$\frac{1}{5} - \frac{1}{1+t^2} = 0 \Rightarrow \frac{1}{5} = \frac{1}{1+t^2}$$

$$1+t^2 = 5$$

$$t^2 = 4 \qquad t = \pm 2$$

Answer:

$t = \pm 2$.

7. (12 points)

(a) Find the linearization of the function $f(x) = \sqrt{x}$ at the point $a = 16$.

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = f(16) + f'(16)(x-16)$$

$$L(x) = \sqrt{16} + \frac{1}{2\sqrt{16}}(x-16)$$

$$L(x) = 4 + \frac{1}{8}(x-16) = \frac{x}{8} + 2$$

Answer:

$$L(x) = \frac{x}{8} + 2$$

(b) Use the linearization to approximate the value of $\sqrt{16.08}$.

$$\sqrt{16.08} = f(16.08) \approx L(16.08) = \frac{16.08}{8} + 2$$

$$= 2.01 + 2$$

$$= 4.01$$

Answer:

$$4.01$$