MTH 141

Midterm 1

May 30, 2022

Name: SOLUTIONS

UR ID: ______________________________

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Instructions:

• The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:______________________________

1. (16 points) Answer each part below and fully justify your answers. Put your final answer in the answer box:

(a) Compute the value of

$$A = \log_3(81) + \log_2\left(\frac{1}{16}\right) - \ln(e^7).$$

$A = 4 + (-4) - 7 = -7$

Answer:

-7

(b) Solve the inequality:

$$|2x - 3| \leq 7.$$

$-7 \leq 2x - 3 \leq 7$

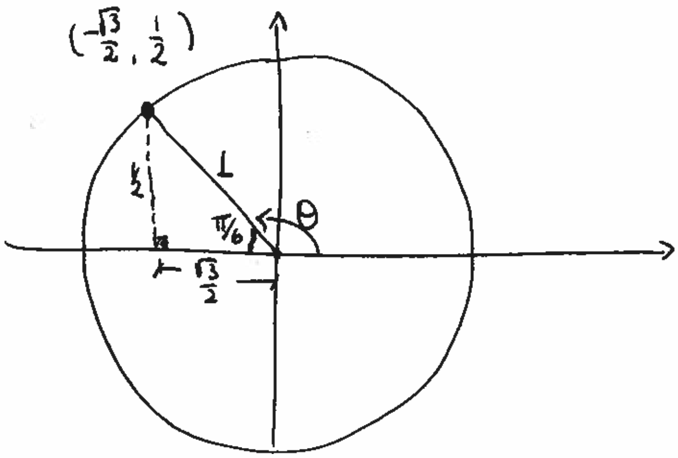
$-4 \leq 2x \leq 10$

$-2 \leq x \leq 5$

Answer:

$[-2, 5]$

(c) If $\sin\theta = 1/2$, and $\pi/2 < \theta < \pi$, compute $\cos\theta$ and $\cot\theta$.



$$\cos\theta = -\frac{\sqrt{3}}{2}$$

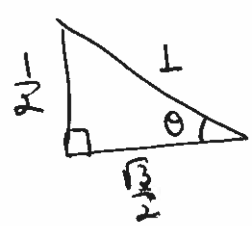
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

Answer:

$\cos\theta = \frac{-\sqrt{3}}{2}$; $\cot\theta = -\sqrt{3}$

(d) Compute $\cos^{-1}(\sqrt{3}/2)$.

$\cos^{-1}(\sqrt{3}/2) = \theta \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$, where $\theta \in [0;\pi]$



$\Rightarrow \theta = \frac{\pi}{6}$

$$\Rightarrow \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Answer:

$\frac{\pi}{6}$

2. (12 points) Let $f(x) = \dfrac{2x-1}{x+2}$. Answer each part below, show your work, and put the final answer in the answer box.

(a) Find an explicit formula for $f^{-1}(x)$.

$$y = \frac{2x-1}{x+2}$$

$$yx + 2y = 2x - 1$$

$$yx - 2x = -1 - 2y$$

$$(y-2)x = -1 - 2y \Rightarrow x = \frac{-1-2y}{y-2} \rightsquigarrow f^{-1}(x) = \frac{-1-2x}{x-2}$$

Answer:

$$f^{-1}(x) = \frac{-1-2x}{x-2} = \frac{2x+1}{2-x}$$

(b) Find the domain and range of f and f^{-1} with brief justification.

Clearly $\mathrm{Dom}(f) = (-\infty, \infty) - \{-2\}$ (since $x+2 \neq 0$)

By (a), $\mathrm{Dom}(f^{-1}) = (-\infty, \infty) - \{2\}$. Then $\mathrm{Ran}(f) = (-\infty, \infty) - \{2\}$

Moreover $\mathrm{Ran}(f^{-1}) = (-\infty, \infty) - \{-2\}$.

(Remember that $\mathrm{Dom}\, f = \mathrm{Ran}\, f^{-1}$ and $\mathrm{Ran}\, f = \mathrm{Dom}\, f^{-1}$)

Answer:

$\mathrm{Dom}(f) = \mathrm{Ran}(f^{-1}) = (-\infty, \infty) - \{-2\}$

$\mathrm{Ran}(f) = \mathrm{Dom}(f^{-1}) = (-\infty, \infty) - \{2\}$

3. (12 points) Let $\mathcal{L}$ be the line which pass for the point $(3, 5)$ and $(-1, -3)$.

(a) Find the equation of the line $\mathcal{L}$.

$$m_L = \frac{-3-5}{-1-3} = \frac{-8}{-4} = 2$$

We have $(3, 5)$ as point and $m_L = 2$

Eq: $y - 5 = 2(x - 3) \Rightarrow y - 5 = 2x - 6$

$y = 2x - 1 \ \dots \ \mathcal{L}$

Answer:

$y = 2x - 1$

(b) Find the equation of the line that goes through the point $(-1, -2)$ and is perpendicular to the line $\mathcal{L}$.

Let m THE SLOPE OF THE PERPENDICULAR LINE OF L.

$\Rightarrow m \cdot m_L = -1 \Rightarrow m \cdot 2 = -1 \Rightarrow m = -\frac{1}{2}$

We have $(-1, -2)$ as point and $m = -\frac{1}{2}$

$\Rightarrow y - (-2) = -\frac{1}{2}(x - (-1)) \Rightarrow y + 2 = -\frac{1}{2}(x + 1)$

$y + 2 = -\frac{x}{2} - \frac{1}{2}$

$y = -\frac{x}{2} - \frac{1}{2} - 2$

$y = -\frac{x}{2} - \frac{5}{2}$

Answer:

$y = -\frac{x}{2} - \frac{5}{2}$

4. (24 points) Evaluate the following limits using limit laws and properties of limits to justify your answers. If they do not exist, explain why not. If the limit is $+\infty$ or $-\infty$, state which it is. Put your final answer in the answer box.

(a) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16}$.

DIFFERENCE OF 2 SQUARES

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x^2-16)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{\cancel{x-4}}{(x+4)\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{(x+4)(\sqrt{x}+2)} = \frac{1}{(4+4)(\sqrt{4}+2)}$$

$$= \frac{1}{8\cdot 4} = \frac{1}{32}$$

Answer:

$\frac{1}{32}$

(b) $\lim_{t\to 0} \frac{\sqrt{t+4}-2}{t}$.

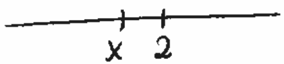
$$\lim_{t\to 0} \frac{\sqrt{t+4}-2}{t} = \lim_{t\to 0} \frac{(\sqrt{t+4}-2)}{t} \cdot \frac{(\sqrt{t+4}+2)}{(\sqrt{t+4}+2)} = \lim_{t\to 0} \frac{t+4-4}{t(\sqrt{t+4}+2)}$$

$$= \lim_{t\to 0} \frac{t}{t(\sqrt{t+4}+2)} = \lim_{t\to 0} \frac{1}{\sqrt{t+4}+2} = \frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$$

Answer:

$\frac{1}{4}$

(c) $\lim_{x \to 2^-} \dfrac{|x-1|-1}{|x-2|}$.



$x < 2 \Rightarrow x - 2 < 0 \Rightarrow |x-2| = -(x-2)$

Moreover x close to 2 $\Rightarrow$ $x-1$ is close to 1 $\Rightarrow$ $x - 1 > 0$

$\Rightarrow |x-1| = x-1.$

Then $\lim_{x \to 2^-} \dfrac{|x-1|-1}{|x-2|} = \lim_{x \to 2^-} \dfrac{x-1-1}{-(x-2)} = \lim_{x \to 2^-} \dfrac{x-\cancel{2}}{-(x-\cancel{2})} = -1$

Answer:

-1

5. (12 points) Consider the following function:

$$f(x) = \begin{cases} x^3 + 5x - 2, & \text{if } x \leq -1 \\ -7, & \text{if } x = -1 \\ \dfrac{(x+1)^2 - 1}{x}, & \text{if } -1 < x < 0 \\ e^x + 1 & \text{if } x \geq 0 \end{cases}$$

(a) Compute each of the following limits or state why they do not exist. Fully justify your answer.

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^3 + 5x - 2 = \cancel{\lim_{x\to -1}} (-1)^3 + 5(-1) - 2 = -1 - 5 - 2 = -8$

x -1

$x < -1$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \dfrac{(x+1)^2 - 1}{x} = \dfrac{(-1+1)^2 - 1}{-1} = \dfrac{0^2 - 1}{-1} = 1$

-1 x

$\lim_{x \to -1} f(x) =$ DNE, since both lateral limits are different

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \dfrac{(x+1)^2 - 1}{x} = \lim_{x \to 0^-} \dfrac{x^2 + 2x + 1 - 1}{x} = \lim_{x \to 0^-} \dfrac{x^2 + 2x}{x} = \lim_{x \to 0} \dfrac{x(x+2)}{x}$

$= \lim_{x \to 0} x + 2 = 2.$

x 0

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^x + 1 = e^0 + 1 = 1 + 1 = 2$

0 x

$\lim_{x \to 0} f(x) = 2$, since both lateral limits are 2.

6. (12 points)

(a) Write the definition that f is continuous at the point $x = b$.

∘) $b \in \mathrm{Dom}(f)$

∘) Exists $\lim_{x \to b} f(x)$

∘) $\lim_{x \to b} f(x) = f(b)$

(b) Consider the following function:

$$f(x) = \begin{cases} c^2x^2 - 5x & \text{if } x < 1, \\ -11\cos(\pi x) + \sin(\pi x) & \text{if } x \geq 1. \end{cases}$$

Find the values of c to make the function f continuous.

We need to compute : $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} -11\cos(\pi x) + \sin(\pi x)$

$= -11 \cos \pi + \sin \pi = 11$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} c^2x^2 - 5x = c^2(1)^2 - 5(1)$

$= c^2 - 5$

We need $\lim_{x \to 1^+} f(x) = \underbrace{\lim_{x \to 1^-} f(x)}_{} = f(1) = 11$

$c^2 - 5 = 11 \Rightarrow c^2 = 16$

$c = \pm 4$

Answer: $c = 4, \quad c = -4$

7. (12 points) Consider the following function:

$$h(x) = \frac{x^2 + x}{x^2 - 4x - 5}.$$

(a) Find all the vertical asymptotes of the function $h(x)$. Justify your answer.

$$h(x) = \frac{x(x+1)}{(x-5)(x+1)}$$

CANDIDATE OF V.A: $x = 5$ and $x = -1$

$$\lim_{x\to 5^+} h(x) = \lim_{x\to 5^+} \frac{x(x+1)}{(x-5)(x+1)} = +\infty \Rightarrow x = 5 \text{ is V. A.}$$

$$\lim_{x\to -1} h(x) = \lim_{x\to -1} \frac{x\cancel{(x+1)}}{(x-5)\cancel{(x+1)}} = \lim_{x\to -1} \frac{x}{x-5} = \frac{-1}{-6} = \frac{1}{6} \Rightarrow x = -1 \text{ is Not V.A}$$

Answer:

$x = 5$

(b) Find all the horizontal asymptotes of the function $h(x)$. Justify your answer.

$$\lim_{x\to +\infty} \frac{x^2+x}{x^2-4x-5} = \lim_{x\to +\infty} \frac{\frac{x^2+x}{x^2}}{\frac{x^2-4x-5}{x^2}} = \lim_{x\to +\infty} \frac{1+\frac{1}{x}}{1-\frac{4}{x}-\frac{5}{x^2}} = \frac{1+0}{1-0-0} = 1$$

$$\lim_{x\to -\infty} \frac{x^2+x}{x^2-4x-5} = \lim_{x\to -\infty} \frac{1+\frac{1}{x}}{1-\frac{4}{x}-\frac{5}{x^2}} = 1$$

Then $y = 1$ is H.A

Answer:

$y = 1.$