MATH 141 Final Exam

December 18, 2022

NAME (please print legibly): Key

Your University ID Number: ______________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________

Enter your answers where indicated in order to receive credit. Calculators and notes are not permitted. If you are confused about the wording of a question or need a clarification, you should raise your hand and **ask a proctor** about it.
Unless otherwise indicated, you must show all work to justify your answers and receive full credit.

Part A

1. (15 points) Determine the following limits. If the limit is infinite, specify $-\infty$ or $+\infty$. Show your work, and justify your answers.

(a) $\lim_{x \to \frac{1}{2}^-} \frac{3x^2+2}{2x-1}$ $= \frac{\text{nonzero}}{\text{zero}} = \pm\infty$

$x \to \frac{1}{2}^-$, so $x < \frac{1}{2}$. Then $2x < 1$ and $2x-1 < 0$.

If $x = \frac{1}{2}$, $3x^2+2 > 0$. Then, as $x \to \frac{1}{2}^-$, $\frac{3x^2+2}{2x-1} = \frac{+}{-} < 0$.

We get $-\infty$

Answer:

$-\infty$

(b) $\lim_{x \to 0} \frac{\sin(x)+1}{\cos(x)+1}$

$= \frac{\sin(0)+1}{\cos(0)+1} = \frac{1}{2}$

Answer:

$\frac{1}{2}$

(c) $\lim_{x \to -3} \left[\frac{4x}{(x-3)(x+3)} - \frac{2}{x+3} \right]$ $= \infty - \infty$

$= \lim_{x \to -3} \frac{4x - 2(x-3)}{(x-3)(x+3)} = \lim_{x \to -3} \frac{4x-2x+6}{(x-3)(x+3)}$

$= \lim_{x \to -3} \frac{2x+6}{(x-3)(x+3)} = \lim_{x \to -3} \frac{2\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \frac{2}{-6}$

Answer:

$\frac{-1}{3}$

2. (9 points)

Suppose $f(x)$ satisfies the inequality

$$2\sqrt{3x} - 4 \leq f(x) \leq x^2 - 5x + 8.$$

State why $\lim_{x\to 3} f(x)$ exists, and find its value.

Since $\lim_{x\to 3} 2\sqrt{3x} - 4 = 2(3) - 4 = 2$

and $\lim_{x\to 3} x^2 - 5x + 8 = 9 - 15 + 8 = 2$,

these outer limits are equal.

Since the inequality holds as well,

we have $\lim_{x\to 3} f(x) = 2$ by the squeeze theorem.

Answer:

2, squeeze theorem holds

3. (12 points) Determine all horizontal and vertical asymptotes of the graph of $y = f(x)$, where

$$f(x) = \frac{\sqrt{x^2 - 6x}}{x+1}.$$

$\boxed{\text{HA}}$

$$\lim_{x\to\infty} \frac{\sqrt{x^2-6x}}{x+1} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(1-\frac{6}{x}\right)}}{x+1} = \lim_{x\to\infty} \frac{\cancel{\sqrt{x^2}}\sqrt{1-\frac{6}{x}}}{\cancel{x}\left(1+\frac{1}{x}\right)} = \frac{\sqrt{1-0}}{1+0} = 1$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{1-\frac{6}{x}}}{x\left(1+\frac{1}{x}\right)} = \lim_{x\to-\infty} (-1)\frac{\sqrt{1-\frac{6}{x}}}{\left(1+\frac{1}{x}\right)} = (-1)\frac{\sqrt{1-0}}{1+0} = -1$$

$\boxed{* \text{ If } x<0,\ \sqrt{x^2} = -x.}$

So the two horizontal asymptotes are $y=1$ and $y=-1$.

$\boxed{\text{VA}}$ Setting the denominator equal to zero, $x+1=0$, so $x=-1$ is possibly a vertical asymptote.

We test $\lim\limits_{x\to-1} \frac{\sqrt{x^2-6x}}{x+1}$ and find that the numerator is non-zero at $x=-1$. Hence $x=-1$ is a vertical asymptote.

HA: $y=1$, $y=-1$

VA: $x=-1$

4. (12 points) Consider the function $f(x) = 2x^3 + 3x^2 + 4$.

(a) Why does the Intermediate Value Theorem (IVT) hold for f on the intervals $[-3, -1]$ and $[-1, 1]$?

f is a polynomial, so continuous on $(-\infty, \infty)$. So it holds on any closed interval.

(b) Does the IVT guarantee a solution for $f(x) = 0$ on the interval $[-3, -1]$? Why or why not?

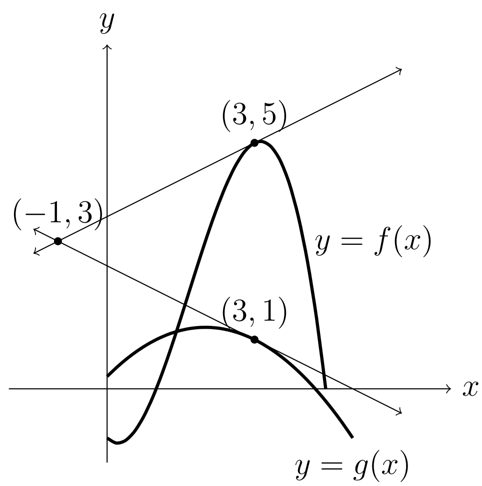
$f(-1) = -2 + 3 + 4 = 5$
$f(-3) = -54 + 27 + 4 = -23$

Since $-23 < 0 < 5$, IVT does guarantee a solution.

(c) Does the IVT guarantee a solution for $f(x) = 0$ on the interval $[-1, 1]$? Why or why not?

$f(1) = 9$. Since $f(-1)$ is also greater than zero, IVT does not guarantee a solution.

5. (12 points) The differentiable functions f and g are graphed below along with their tangent lines at $x = 3$.



(a) Compute $f'(3)$.

The slope of the tangent at $(3, f(3))$ is $\frac{5-3}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$.

Hence $f'(3) = \frac{1}{2}$.

Answer: $\frac{1}{2}$

(b) Compute $g'(3)$.

The slope of the tangent at $(3, g(3))$ is $\frac{1-3}{3-(-1)} = \frac{-2}{4}$.

Hence $g'(3) = \frac{-1}{2}$

Answer: $-\frac{1}{2}$

(c) Let $h(x) = f(x)g(x)$. Compute $h'(3)$.

$h'(x) = f(x)g'(x) + f'(x)g(x)$

$h'(3) = f(3)g'(3) + f'(3)g(3)$

$= (5)(-\frac{1}{2}) + (\frac{1}{2})(1)$

$= -\frac{5}{2} + \frac{1}{2} = -\frac{4}{2} = -2$

Answer: -2

6. (10 points) Let $f(x) = e^{2x+1}$. Determine $f^{-1}(x)$, the inverse function.

$$y = e^{2x+1}$$

$$\ln y = \ln(e^{2x+1}) = 2x+1$$

$$\ln y - 1 = 2x$$

$$x = \frac{\ln y - 1}{2}$$

$$f^{-1}(x) = \frac{\ln(x) - 1}{2}$$

7. (20 points) Find the derivatives of the following functions. You do not need to simplify. Circle your answer.

(a) $g(t) = \dfrac{e^t \sin t}{t^2}$

$$g'(t) = \frac{t^2(e^t \cos t + e^t \sin t) - e^t \sin t\,(2t)}{t^4}$$

(b) $h(r) = (r^2 - 2)^5 \tan(2r + 3)$

$$h'(r) = (r^2-2)^5 \sec^2(2r+3)(2) + 5(r^2-2)(2r)\tan(2r+3)$$

(c) $f(x) = e^{e^x}$

$f'(x) = e^{e^x} e^x$

(d) $k(T) = \ln(2T)3^T$

$k'(T) = \ln(2T)(3^T)(\ln 3) + \frac{2}{2T}(3^T)$

8. (10 points) Use **logarithmic differentiation** to find the derivative of y.

$$y = \frac{e^{2x}(x^2+1)^5(x^6+2)}{(1+e^x)x^4}.$$

$$\ln y = \ln\left[e^{2x}(x^2+1)^5(x^6+2)\right] - \ln\left[(1+e^x)x^4\right]$$

$$= \ln(e^{2x}) + \ln\left[(x^2+1)^5\right] + \ln(x^6+2) - \ln(1+e^x) - \ln(x^4)$$

$$= 2x + 5\ln(x^2+1) + \ln(x^6+2) - \ln(1+e^x) - 4\ln x$$

$$\frac{y'}{y} = 2 + \frac{5(2x)}{x^2+1} + \frac{6x^5}{x^6+2} - \frac{e^x}{1+e^x} - \frac{4}{x}$$

$$y' = \frac{e^{2x}(x^2+1)^5(x^6+2)}{(1+e^x)x^4}\left[2 + \frac{10x}{x^2+1} + \frac{6x^5}{x^6+2} - \frac{e^x}{1+e^x} - \frac{4}{x}\right]$$

Part B

9. (15 points)

The position of a particle moving along the real number line is given by

$$s(t) = t^3 - 9t^2 + 24t - 10.$$

(a) The particle changes direction exactly once between $t = 0$ and $t = 3$. Determine when this takes place.

Set $s'(t) = 0$.

$3t^2 - 18t + 24 = 0$

$3(t^2 - 6t + 8) = 0$

$3(t-4)(t-2) = 0$

$t = 2, 4$

If $t < 2$, then $s'(t) > 0$, so moving forward.

If $2 < t < 4$, then $s'(t) < 0$, so moving backward.

Then the particle changes direction when $t = 2$.

(b) Compute the total distance traveled by the particle between $t = 0$ and $t = 3$.

$s(0) = -10$.

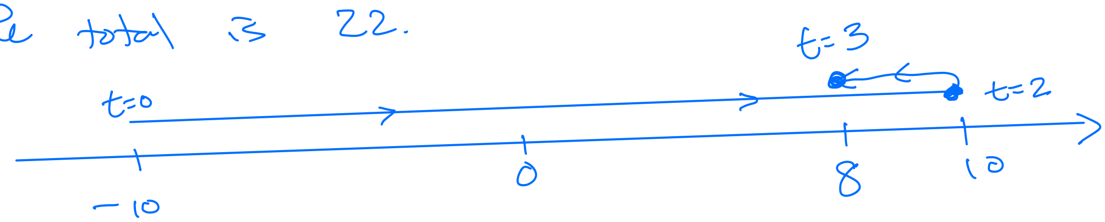
$s(2) = 8 - 9(4) + 24(2) - 10 = 10$

So the particle travels 20 units in the first 2 seconds.

$s(3) = 27 - 81 + 24(3) - 10 = 8$

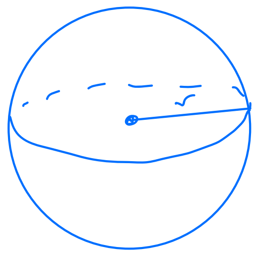
So the particle travels 2 units in the third second.

The total is 22.



10. (20 points) Suppose a spherical balloon is being filled with air so that its volume is increasing at a rate of 16 cm^3/minute. (The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$. The surface area of a sphere is given by $S = 4\pi r^2$.)

(a) When the radius of the balloon is 2 cm, at what rate is the radius changing?



$\frac{dV}{dt} = 16$ Find $\frac{dr}{dt}$ when $r = 2$.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi (3r^2) \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$= 16\pi \frac{dr}{dt}$ if $r = 2$.

Then $\frac{dr}{dt} = \frac{16}{16\pi} = \frac{1}{\pi}$ cm/min

(b) When the radius of the balloon is 2 cm (as in part (a)), at what rate is the surface area of the balloon changing?

$S = 4\pi r^2$

$\frac{dS}{dt} = 4\pi (2r) \frac{dr}{dt}$

$= 4\pi (4)(\frac{1}{\pi}) = 16$ cm^2/min

11. (10 points) Use linearization to estimate $\sqrt[3]{8.12}$. Your answer should be in decimal form.

$L(x) = f(a) + f'(a)(x-a)$.

Let $a = 8$ and $f(x) = \sqrt[3]{x}$.

Then $f(a) = \sqrt[3]{8} = 2$, and $f'(x) = \frac{1}{3}x^{-2/3}$.

$f'(a) = \frac{1}{3}\, 8^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$.

$L(x) = 2 + \frac{1}{12}(x-8)$.

$L(8.12) = 2 + \frac{1}{12}(8.12-8)$

$= 2 + \frac{1}{12}(.12)$

$= 2.01$

12. (10 points) Find the absolute minimum and maximum values of the function

$$f(x) = x^2 e^x$$

on the interval $[-1, 1]$.

$$f'(x) = x^2 e^x + 2x e^x$$
$$= e^x (x^2 + 2x).$$

To find critical numbers, set $e^x(x^2+2x) = 0$.

Since $e^x \neq 0$, critical numbers satisfy $x(x+2) = 0$.

So $x = 0, -2$ are the critical numbers.

We'll ignore $x = -2$, as it isn't in the interval.

Testing f at $x = 0$ and the endpoints of the interval:

x	$f(x)$
0	$(0)(1) = 0$
-1	$1e^{-1} = \frac{1}{e}$
1	$e^1 = e$

Then the absolute min value is 0, and it occurs at $x = 0$.

The absolute max value is e, and it occurs at $x = 1$.

13. (10 points) Let $f(x) = 6 - 3x^2$.

Determine the number c in the interval $[-2, 1]$ guaranteed by the Mean Value Theorem.

MVT say that there is a c in $(-2,1)$ such that $f'(c) = \dfrac{f(1) - f(-2)}{1-(-2)}$. This is true, because f is continuous and differentiable on all of $(-\infty, \infty)$, as it is a polynomial.

$f(1) = 6 - 3 = 3$

$f(-2) = 6 - 3(4) = -6$

$$\frac{f(1) - f(-2)}{1-(-2)} = \frac{3-(-6)}{1-(-2)} = \frac{9}{3} = 3.$$

$f'(x) = -6x.$

Setting $-6x = 3$

$x = -\frac{1}{2}.$

So $c = -\frac{1}{2}$ is the desired number.

14. (15 points) Determine the following limits.

(a) $\lim_{t\to 0} \frac{e^t - e^{2t}}{t}$ This limit has the indeterminate form $\frac{0}{0}$.

By L'Hopital's rule,

$$\lim_{t\to 0} \frac{e^t - e^{2t}}{t} = \lim_{t\to 0} \frac{e^t - 2e^{2t}}{1} = e^0 - 2e^0 = 1-2=-1.$$

(b) $\lim_{x\to 0^+} x^{2x}$ This limit has the indeterminate form 0^0.

Set $y = x^{2x}$. Then $\ln y = 2x \ln x$.

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} 2x\ln x = \lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}}$$

This limit has the indeterminate form $\frac{-\infty}{\infty}$.

By L'Hopital's Rule,

$$\lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{2}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{2}{x}\cdot\frac{-x^2}{1} = \lim_{x\to 0^+} -x = 0.$$

Finally,

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} \ln y} = e^0 = 1.$$

15. (8 points) Suppose $f(x)$ is a **polynomial**.

(a) Suppose $f(4)$ is a local (relative) minimum value. Which of the following are ***possible***? Choose either "Possible" or "Not Possible" in each case.

(i) $f'(4) = 0$ and $f''(4) > 0$. **Possible** **Not Possible**

Ex $f(x) = (x-4)^2$

(ii) $f'(4) = 0$ and $f''(4) < 0$. **Possible** **Not Possible**

If $f''(4) < 0$, then f is concave down at $x = 4$.

(iii) $f'(4) = 0$ and $f''(4) = 0$. **Possible** **Not Possible**

Ex $f(x) = (x-4)^3$

(iv) $f'(4) < 0$ and $f''(4) = 0$. **Possible** **Not Possible**

If $f'(4) < 0$, f is decreasing at $x = 4$, and it cannot have a local min there.

(b) Suppose $(2, f(2))$ is an inflection point. Which of the following are ***possible***? Choose either "Possible" or "Not Possible" in each case.

(i) $f'(2) = 0$ and $f''(2) = 0$. **Possible** **Not Possible**

Ex: $f(x) = (x-2)^3$

(ii) $f'(2) = 0$ and $f''(2) < 0$. **Possible** **Not Possible**

If $f''(2) < 0$, f is concave down at $x = 2$, so it can't have an inflection point there.

(iii) $f'(2) > 0$ and $f''(2) = 0$. **Possible** **Not Possible**

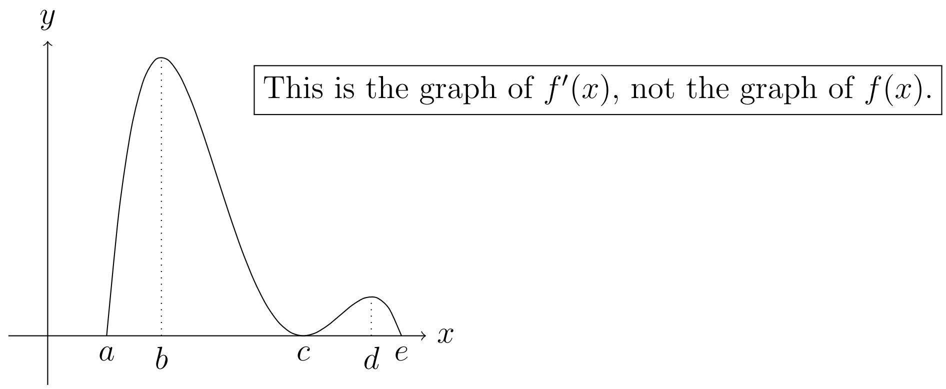
Ex $f(x) = -(x-4)e^{x-4}$.

This is a reflection and translation of the more familiar $f(x) = xe^x$.

(iv) $f'(2) < 0$ and $f''(2) > 0$. **Possible** **Not Possible**

Similar to (ii).

16. (12 points) Suppose an unknown function f is differentiable on (a, e) and that below is the graph of its ***derivative*** on that interval.



(a) Put the values $f'(b), f'(c), f'(d)$ in order from least to greatest.

Answer: $f'(c)$ < $f'(d)$ < $f'(b)$

(b) Put the values $f(b), f(c), f(d)$ in order from least to greatest.

Since $f'(x) \geq 0$, it increases.

Answer: $f(b)$ < $f(c)$ < $f(d)$

(c) On the interval (b, c), is $f''(x)$ (the second derivative) positive or negative? (Justify your answer.)

It is negative. Since $f'(x)$ decreases on that interval, its derivative is negative.

(d) Determine any points of inflection of the graph of f. (Justify your answer.)

$(b, f(b))$, $(c, f(c))$, and $(d, f(d))$ are inflection points.

We can determine this, because f' changes from increasing to decreasing to increasing, etc., so f'' changes sign at those points.