MTH 141

Final exam

June 22, 2023

Name: SOLUTIONS

UR ID: ______________________

Instructor's Name: Andrés Chirre

Instructions:

• The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• We will judge your work outside the box as well (unless specified otherwise) so you still need to show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ______________________

QUESTION	VALUE	SCORE
1	6	
2	6	
3	8	
4	9	
5	7	
6	6	
7	8	
8	8	
9	6	
10	6	
11	6	
12	6	
13	12	
14	6	
TOTAL	100	

Some formulas

- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cot^2\theta + 1 = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
- $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
- $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
- $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$
- $(fg)' = f'g + fg'$; $\quad \left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$; $\quad \lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$; $\quad \lim\limits_{h\to 0}\dfrac{\cos h - 1}{h} = 0$

Part A

1. (6 points) For this problem, justification is not required and partial credit will NOT be awarded. Fill in where you see ???.

(a) Let $y = mx + b$ be the equation of the line that goes through the point $(1, 2)$ and is perpendicular to the line $3x - 7y = 1$. Then $b =$???

Answer:

$b = \frac{13}{3}$

(b) $\cos(\arctan(\sqrt{3}$/[redacted]$))$ $=$???

Answer:

$\frac{1}{2}$

(c) Let $f(x) = x^2 + 2x - 2$, and $g(x) = \sqrt{x-1}$. Then $(f \circ g)(5) =???$

Answer:

6

2. (6 points) Let $g(x) = \dfrac{2x-3}{x+5}$. Justify your work in the following questions.

(a) Find an explicit formula for $g^{-1}(x)$.

$$y = \frac{2x-3}{x+5} \Rightarrow yx + 5y = 2x - 3$$

$$yx - 2x = -3 - 5y$$

$$(y-2)x = -3 - 5y$$

$$x = \frac{-3-5y}{y-2} \Rightarrow g^{-1}(x) = \frac{-3-5x}{x-2}$$

Answer:

$$g^{-1}(x) = \frac{-3-5x}{x-2} = \frac{3+5x}{2-x}$$

(b) Compute $g^{-1}(3)$.

$$g^{-1}(3) = \frac{3+5(3)}{2-3} = \frac{18}{-1} = -18$$

Answer:

-18

3. (8 points) In the following questions, justify your answer by using properties/theorems involving limits. Consider the function:

$$f(x) = \frac{3x-3}{2x^2-6x+4}.$$

(a) Compute $\lim_{x\to 1} f(x)$ or justify why such limit does not exist. (NO points will be awarded if l'Hospital's rule is used.)

$$f(x) = \frac{3(x-1)}{2(x^2-3x+2)} = \frac{3(x-1)}{2(x-2)(x-1)} = \frac{3}{2(x-2)}, \quad \text{when } x \neq 1$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{3}{2(x-2)} = \frac{3}{2(1-2)} = -\frac{3}{2}$$

Answer:

$-\frac{3}{2}$

(b) List the vertical asymptote(s) of $f(x)$. Justify your work properly.

The candidate to be vertical asymptote are $x=2$ and $x=1$.

But, since $\lim_{x\to 1} f(x) = -\frac{3}{2}$, then $x=1$ is NOT A VERTICAL ASYMPTOTE.

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{3}{2(x-2)} = +\infty$, then $x=2$ is A VERTICAL ASYMPTOTE.

*) Note that you can justify also by $\lim_{x\to 2^-} f(x) = -\infty$, but NO WITH "$\lim_{x\to 2} f(x)$" because this limit DNE.

Answer:

$x=2$

4. (9 points) For each part, show your work and put your answer in the answer box.

(a) Compute the following limit

$$\lim_{x\to+\infty}(\sqrt{x^2+2}-x).$$

$$\lim_{x\to\infty}\sqrt{x^2+2}-x=\lim_{x\to+\infty}\left(\sqrt{x^2+2}-x\right)\frac{\left(\sqrt{x^2+2}+x\right)}{\left(\sqrt{x^2+2}+x\right)}$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+2}\right)^2-x^2}{\sqrt{x^2+2}+x}=\lim_{x\to\infty}\frac{x^2+2-x^2}{\sqrt{x^2+2}+x}$$

$$=\lim_{x\to\infty}\frac{2}{\sqrt{x^2+2}+x}=0$$

Answer:

0

(b) Compute the limit

$$\lim_{t\to5}\frac{t-5}{\sqrt{t-1}-2}.$$

$$\lim_{t\to5}\frac{t-5}{\sqrt{t-1}-2}=\lim_{t\to5}\frac{t-5}{(\sqrt{t-1}-2)}\cdot\frac{(\sqrt{t-1}+2)}{(\sqrt{t-1}+2)}$$

$$=\lim_{t\to5}\frac{(t-5)(\sqrt{t-1}+2)}{\left(\sqrt{t-1}\right)^2-2^2}=\lim_{t\to5}\frac{(t-5)(\sqrt{t-1}+2)}{t-5}$$

$$=\lim_{t\to5}\sqrt{t-1}+2=4$$

Answer:

4

(c) Consider the function $f(x)$ which satisfies the inequality

$$\sqrt[3]{x+9} \leq f(x) \leq x^2 + 2x + 3.$$

Compute $\lim_{x \to -1} f(x)$, and justify why this limit exists.

Note that $\lim_{X \to -1} \sqrt[3]{X+9} = \sqrt[3]{8} = 2$

and

$\lim_{X \to -1} X^2 + 2X + 3 = (-1)^2 + 2(-1) + 3 = 2$

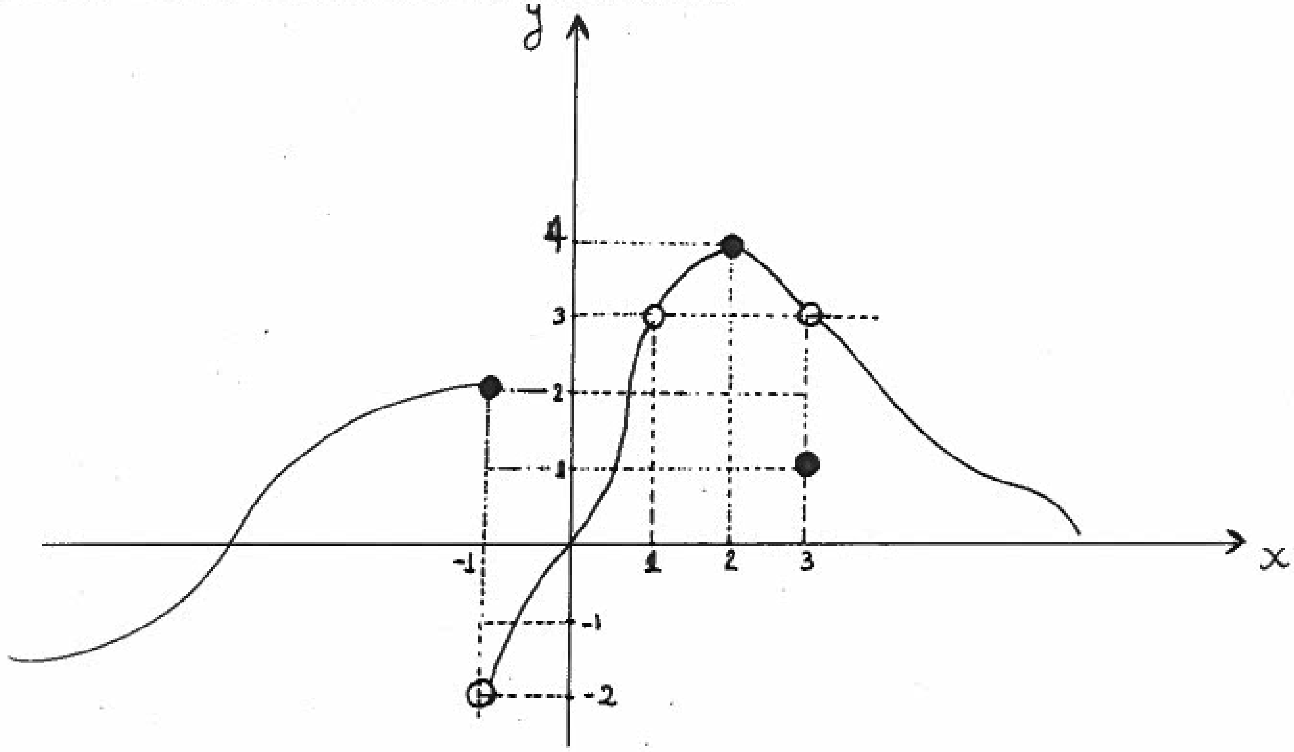
Then, by Squeez's theorem we have that the limit exists

and $\lim_{X \to -1} f(X) = 2$

Answer:

2

5. (7 points) Let f be a function and consider its graphic:



Write TRUE or FALSE in each ( ). Justification is not required and partial credit will NOT be awarded

(a) f is continuous at $x = 1$ .... (FALSE)

(b) $\lim_{x \to -1} f(x) = 2$ .... (FALSE)

(c) $\lim_{x \to 2^+} f(x) = 4$ .... (TRUE)

(d) $\lim_{x \to 3^-} f(x) = 3$ .... (TRUE)

(e) $\lim_{x \to 1^-} f(x)$ does not exist .... (FALSE)

(f) $\lim_{x \to 3} f(x) = f(3)$ .... (FALSE)

(g) f is continuous at $x = 3$ .... (FALSE)

6. (6 points)

(a) Let $f(x) = \dfrac{1}{2+x}$, and recall the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Use the definition to find $f'(x)$.
Note: NO points will be awarded if you do not use the definition or if you use l'Hospital's rule.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{2+x+h} - \frac{1}{2+x}}{h} = \lim_{h \to 0} \frac{\frac{2+x-(2+x+h)}{(2+x+h)(2+x)}}{h} = \lim_{h \to 0} \frac{-h}{h(2+x+h)(2+x)}$$

$$= \lim_{h \to 0} \frac{-1}{(2+x+h)(2+x)} = \frac{-1}{(2+x)^2}$$

Answer:

$$f'(x) = \frac{-1}{(2+x)^2}$$

(b) Compute the equation of the line tangent to $y = f(x)$ at $x = 1$.

$$m = f'(1) = \frac{-1}{(2+1)^2} = \frac{-1}{9}$$

POINT:

$$(1,\ f(1)) = (1,\ \tfrac{1}{3}) \Rightarrow y - \frac{1}{3} = -\frac{1}{9}(x-1)$$

$$y - \frac{1}{3} = -\frac{x}{9} + \frac{1}{9} \Rightarrow y = -\frac{x}{9} + \frac{4}{9}$$

Answer:

$$y = -\frac{x}{9} + \frac{4}{9}$$

7. **(8 points)** Find the derivatives of the following functions. Show your work and put your answer in the answer box.

(a) $f(x) = (\pi^2 + \pi^x)e^x - \sqrt[3]{x^2 - 2}$.

$$f(x) = (\pi^2 + \pi^x)\, e^x - (x^2-2)^{\frac{1}{3}}$$

$$f'(x) = (\pi^2+\pi^x)'\, e^x + (\pi^2+\pi^x)\, e^x - \frac{1}{3}(x^2-2)^{-\frac{2}{3}} . (x^2-2)'$$

$$f'(x) = (\pi^x \ln\pi)e^x + (\pi^2+\pi^x)\, e^x - \frac{2x}{3(x^2-2)^{\frac{2}{3}}}$$

Answer:

$$f'(x) = \pi^x e^x \ln\pi + (\pi^2+\pi^x)\, e^x - \frac{2x}{3(x^2-2)^{2/3}}$$

(b) $g(x) = e^{\tan(x) + (\ln x)^2}$.

$$g'(x) = e^{\tan x + (\ln x)^2} \cdot \left[\tan x + (\ln x)^2\right]'$$

$$= e^{\tan x + (\ln x)^2} \cdot \left[\sec^2 x + 2\ln x . \frac{1}{x}\right]$$

Answer:

$$g'(x) = e^{\tan x + (\ln x)^2}\left[\sec^2 x + \frac{2\ln x}{x}\right]$$

(c) $f(x) = \arctan(\sqrt{x^4+16})$.

$f(x) = \arctan\left(\left(x^4+16\right)^{1/2}\right)$

$$f'(x) = \frac{1}{1+\left(\sqrt{x^4+16}\right)^2} \cdot \frac{1}{2}(x^4+16)^{-\frac{1}{2}} \cdot 4x^3$$

$$f'(x) = \frac{1}{x^4+17} \cdot 2\,(x^4+16)^{-\frac{1}{2}}\,x^3$$

Answer:

$$f'(x) = \frac{2x^3}{(x^4+17)\sqrt{x^4+16}}$$

(d) $g(x) = e^{e^x}$.

$e^x = u \rightarrow \frac{du}{dx} = e^x$

$g(u) = e^u \Rightarrow \frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = e^u \cdot e^x$

$$\frac{dg}{dx} = e^{e^x} \cdot e^x$$

Answer:

$$g'(x) = e^{e^x} \cdot e^x$$

Part B

8. (8 points) Consider the equation

$$y^2 + 2\arctan(y) + \tan x = 2x.$$

(a) Use implicit differentiation to compute $\dfrac{dy}{dx}$.

By implicit diffrentiation

$$2yy' + \frac{2}{1+y^2} \cdot y' + \sec^2 x = 2$$

$$y'\left(2y + \frac{2}{1+y^2}\right) = 2 - \sec^2 x \Rightarrow y' = \frac{2-\sec^2 x}{2y + \frac{2}{1+y^2}}$$

Answer:

$$\frac{dy}{dx} = \frac{2-\sec^2 x}{2y + \frac{2}{1+y^2}}$$

(b) Find the equation of the tangent line at the point $(0, 0)$.

$$m = \frac{2-\sec^2 0}{2\cdot 0 + \frac{2}{1+0^2}} = \frac{2-1}{0+2} = \frac{1}{2}$$

POINT $(0,0)$ $\qquad y - 0 = \frac{1}{2}(x-0) \Rightarrow y = \frac{x}{2}$

Answer:

$$y = x/2$$

9. **(6 points)** Use linearization to approximate the value of $\sqrt[3]{8.1}$. Justify your work.

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$$

$a = 8$

LINEARIZATION:

$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt[3]{8} + \frac{1}{3 \cdot 8^{\frac{2}{3}}}(x-8)$$

$$= 2 + \frac{1}{3 \cdot 4}(x-8)$$

$$L(x) = 2 + \frac{x}{12} - \frac{2}{3} = \frac{4}{3} + \frac{x}{12}$$

$$L(x) = \frac{16+x}{12}$$

$$f(8.1) \approx L(8.1) = \frac{16+8.1}{12} = 2{,}00833\ldots$$

Answer:

$2{,}00833\ldots$ or $\frac{24.1}{12}$

10. (6 points) Use logarithmic differentiation to compute the derivative of

(a)

$$f(x) = x^{e^x}.$$

$\ln f(x) = \ln(x^{e^x}) = e^x \ln x$

$\frac{f'(x)}{f(x)} = e^x . \ln x + e^x . \frac{1}{x}$

$\Rightarrow f'(x) = \left(e^x \ln x + \frac{e^x}{x}\right) x^{e^x}$

Answer:

$f'(x) = \left(e^x \ln x + \frac{e^x}{x}\right) x^{e^x}$

(b)

$$g(x) = (\sqrt{x})^{\arctan x}.$$

$\ln g(x) = \arctan x \ln(\sqrt{x}) = \arctan x . \frac{1}{2} \ln x$

$\frac{g'(x)}{g(x)} = \frac{1}{2} \cdot \left[\arctan x . \ln x\right]' = \frac{1}{2}\left[\frac{1}{1+x^2} . \ln x + \arctan x . \frac{1}{x}\right]$

$g'(x) = \left(\frac{\ln x}{2(1+x^2)} + \frac{\arctan x}{2x}\right)(\sqrt{x})^{\arctan x}$

Answer:

$g'(x) = \left(\frac{\ln x}{2(1+x^2)} + \frac{\arctan x}{2x}\right)(\sqrt{x})^{\arctan x}$

11. (6 points) Find the absolute maximum and absolute minimum of the function $f(x) = x^2e^x$ on the interval $[-2, 2]$. Justify your work.

We use the closed interval method.

1°) FIND THE CRITICAL NUMBERS IN $(-2, 2)$:

$$f'(x) = 2xe^x + x^2e^x$$

$$f'(x) = e^x \cdot x(2+x)$$

$$f'(x) = 0 \Rightarrow x = 0, \; x = -2.$$

CRITICAL NUMBER "$x = 0$".

2°) EVALUATE THE CRITICAL NUMBER ~~AT~~ $x = 0$:

$$f(0) = 0^2e^0 = 0.$$

3°) EVALUATE ~~THE~~ ~~[illegible]~~ END POINTS $x = -2$, $x = 2$

$$f(-2) = (-2)^2e^{-2} = 4/e^2$$

$$f(2) = 2^2e^2 = 4e^2$$

4°) COMPARE: LARGEST VALUE: $4e^2 = f(2)$.

SMALLEST VALUE: $0 = f(0)$

Answer:

ABSOLUTE MAXIMUM: $4e^2 = f(2)$.

ABSOLUTE MINIMUM: $0 = f(0)$.

12. (6 points) Let f be a function.

(a) Complete the statement of the mean value theorem.

I) f is ...CONTINUOUS........ on the closed interval $[a, b]$.

II) f is DIFFERENTIABLE... on the open interval (a, b).

Then, there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) Suppose that $f(-1) = -1$ and $f'(x) \leq 5$. How large can $f(2)$ possibly be? Justify your answer.

USING M.V.T:

$$\frac{f(2) - f(-1)}{2 - (-1)} = f'(c) \quad , \quad c \in (-1, 2)$$

$$f(2) - (-1) = 3f'(c) \Rightarrow f(2) = 3f'(c) - 1$$

But $f'(x) \leq 5$, then $f(2) = 3f'(c) - 1 \leq 3.5 - 1$

$$f(2) \leq 14$$

Answer: 14.

13. (12 points) Consider the function $f(x) =$ ~~[illegible]~~. $2x^3 - 3x^2 - 36x$.

(a) Find the intervals of increase or decrease.

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$f'(x) = 6(x-3)(x+2)$

f'

f INCREASING $(-\infty, -2) \cup (3, +\infty)$

f DECREASING $(-2, 3)$

Answer:

(b) Find the local maximum and local minimum values (if they exist).

f'

$X = -2$ HAS A LOCAL MAXIMUM

CHANGE OF SIGN

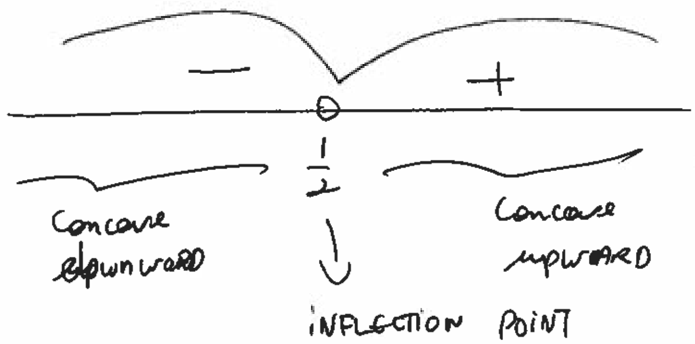
$X = 3$ HAS A LOCAL MINIMUM.

Answer:

(c) Find the intervals where the function is concave upward and concave downward, and the inflection points.

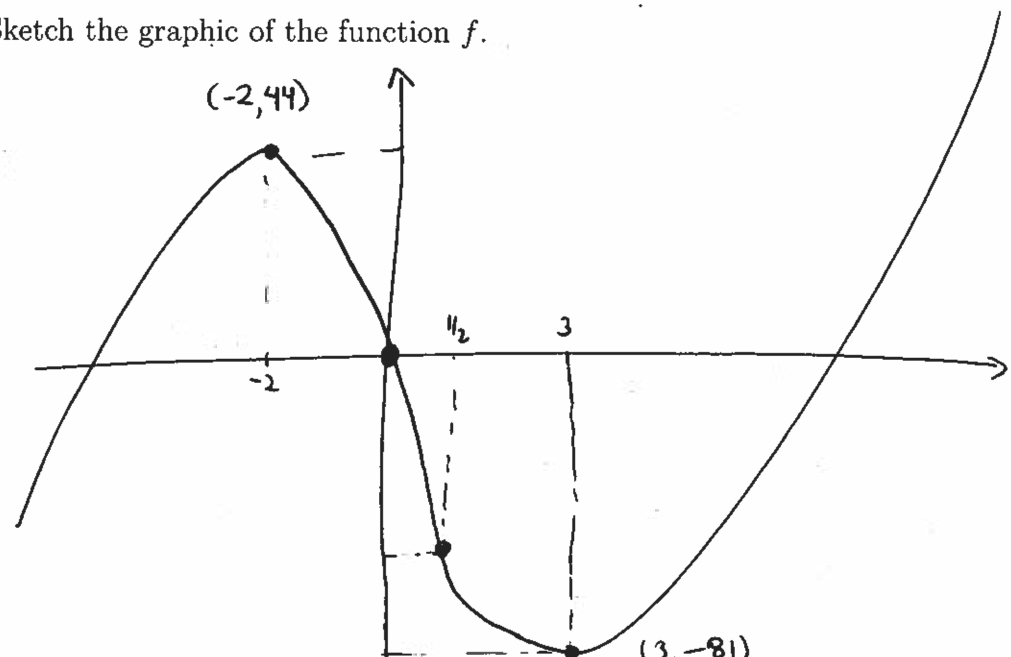
$f'(x) = 6x^2 - 6x - 36$

$f''(x) = 12x - 6 = 6(2x-1)$



Answer:
IN $(-\infty; \frac{1}{2})$, f is concave downward,
IN $(\frac{1}{2}, +\infty)$; f is concave upward.
$x = \frac{1}{2}$ is AN INFLECTION POINT.

(d) Sketch the graphic of the function f.



Answer:

14. (6 points) .In each case, use l'Hospital's rule to compute the limit. Justify your work and your answer.

(a)

$$\lim_{x\to+\infty} \frac{3e^{2x}}{5x^2+x-4}$$

IT HAS IND. FORM "$\frac{\infty}{\infty}$", then by l'HOSPITAL'S RULE

again IT is IND FORM "$\frac{\infty}{\infty}$", then by l' HOSPITAL'S RULE

$$\lim_{x\to\infty} \frac{3e^{2x}}{5x^2+x-4} = \lim_{x\to+\infty} \frac{6e^{2x}}{10x+1} = \lim_{x\to\infty} \frac{12e^{2x}}{10} = +\infty$$

Answer:

$+\infty$

(b)

$$\lim_{x\to 0^+} (7x+1)^{1/x}$$

$$\lim_{x\to 0^+} (7x+1)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\ln\left((7x+1)^{\frac{1}{x}}\right)} = \lim_{x\to 0^+} e^{\frac{1}{x}\cdot\ln(7x+1)}$$

$$= e^{\lim_{x\to 0^+} \frac{\ln(7x+1)}{x}}$$

Now, $\lim_{x\to 0^+} \frac{\ln(7x+1)}{x} \overset{\text{IND FORM "0/0", BY HOSPITAL}}{=} \lim_{x\to 0^+} \frac{\frac{7}{7x+1}}{1} = 7$. Then $\lim_{x\to 0^+} (7x+1)^{\frac{1}{x}} = e^7$.

Answer:

e^7