MTH 141

Final Exam

December 13, 2020, 9PM Version

Before we begin, please copy the following honor pledge and sign your name. (The same honesty standards apply as on the midterms. Cursive is not required).

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

TO RECEIVE FULL CREDIT, JUSTIFY YOUR ANSWERS.

Please tell us your instructor's name.

Please tell us your proctor's name.

This information will be submitted at the end of the exam as Question 10.

1. (12 points) Evaluate the following expressions.

(V1)

(a) $e^{3\ln 2}$

$= e^{\ln(2^3)} = e^{\ln 8} = 8$

(b) $\log_2 12 + \log_2 14 - \log_2 21$

$= \log_2 \frac{(12)(14)}{21} = \log_2 \frac{(\cancel{3}\cdot 4\cdot \cancel{7}\cdot 2)}{\cancel{3}\cdot\cancel{7}} = \log_2 (8) = 3$

(c) $\sec^2(\pi/4) - \csc(\pi/2)$

$= \frac{1}{(\cos(\frac{\pi}{4}))^2} - \frac{1}{\sin(\frac{\pi}{2})} = 2 - 1 = 1$

(d) $\tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$

(V2)

(a) $e^{3\ln 2}$

$= 8$

(b) $\log_7 98 + \log_7 1 - \log_7 2$

$= \log_7 \frac{98}{2} + \log_7(1) = \log_7(49) + 0 = 2$

(c) $\sec^2(2\pi/3) - \cot(\pi/4)$

$= 4 - 1 = 3$

(d) $\tan^{-1}(-\sqrt{3})$

$= -\frac{\pi}{3}$

2. (8 points)

(V1) Let

$$f(x) = \frac{x(x^2-16)(2x-1)}{(-3x+2)(x+4)(x^2+1)}.$$

(a) Find all horizontal asymptotes of the graph $y = f(x)$.

(b) Find all vertical asymptotes of the graph $y = f(x)$.

(c) Find all numbers x at which $f(x)$ is discontinuous. Categorize each discontinuity as a removable, jump, or infinite discontinuity.

(a) The highest power term in the numerator is $2x^4$. In the denominator, it's $-3x^4$.

Then $\lim_{x\to\infty} f(x) = \dfrac{x^4\,(2 + \text{terms with limit zero})}{x^4\,(-3 + \text{terms with limit zero.})} = -\dfrac{2}{3}$

$\lim_{x\to-\infty} f(x)$ is the same.

(b) The zeros of the denominator are $x = \frac{2}{3}$ and $x = -4$, as $x^2+1 \geq 1$. We note that $x=-4$ is also a zero of the numerator.

$$\lim_{x\to 4} f(x) = \lim_{x\to -4} \frac{x(x-4)\cancel{(x+4)}(2x-1)}{(-3x+2)\cancel{(x+4)}(x^2+1)} = \frac{(-4)(-8)(-9)}{(14)(17)},$$

which is finite. Hence there is no VA @ $x=-4$.

Since $\frac{2}{3}$ is not a zero of the numerator, we have a VA @ $x = \frac{2}{3}$.

(c) A rational function is continuous on its domain, so we need only consider $x=-4$ and $x = \frac{2}{3}$. From (b), there is a removable discontinuity at $x=-4$, and an infinite one at $x = \frac{2}{3}$.

2. (8pts)

(V2) Let

$$f(x) = \frac{x(x-1)(7x-2)}{(-3x+2)(x^2-1)}.$$

(a) Find all horizontal asymptotes of the graph $y = f(x)$.

(b) Find all vertical asymptotes of the graph $y = f(x)$.

(c) Find all numbers x at which $f(x)$ is discontinuous. Categorize each discontinuity as a removable, jump, or infinite discontinuity.

Similar to V1. Removable @ $x=1$ and infinite @ $x=-1$ and $x = \frac{2}{3}$.

3. (6 points)

(V1) Let $f(x) = 7 + \ln(x+6)$.

(a) Find $f^{-1}(x)$, the inverse function of f.

(b) Find the domain and range of f.

$y = 7 + \ln(x+6)$

$y - 7 = \ln(x+6)$

$e^{y-7} = x+6$

$x = e^{y-7} - 6$

$f^{-1}(x) = e^{x-7} - 6$

domain: $(-\infty, \infty)$

range: $(-6, \infty)$

You can also determine domain and range by considering the range and domain of f.

(V2) Let $f(x) = e^{x+4} - 3$.

(a) Find $f^{-1}(x)$, the inverse function of f.

(b) Find the domain and range of f.

$y = e^{x+4} - 3$

$y + 3 = e^{x+4}$

$\ln(y+3) = x+4$

$x = \ln(y+3) - 4$

$f^{-1}(x) = \ln(x+3) - 4$

domain: $(-3, \infty)$

range: $(-\infty, \infty)$

As above, double check by considering range and domain of f.

4. (8 points)

(a) Solve for x:

$$2x - 1 = |13 + 5x|$$

Case 1

If $13 + 5x > 0$

$2x - 1 = 13 + 5x$

$-3x = 14$

$x = \frac{-14}{3}$

Case 2

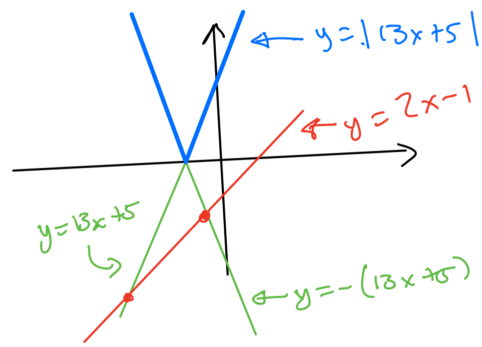
If $13 + 5x < 0$

$2x - 1 = -(13 + 5x)$

$7x = -12$

$x = \frac{-12}{7}$

Neither of these is a solution, however, as $2x - 1 < 0$ in either case.



(b) (V1) Solve for x:

$$5^{2x-1} = 25.$$

$5^{2x-1} = 5^2$

$2x - 1 = 2$

$2x = 3$

$x = \frac{3}{2}$

(b)(V2) Solve for x:

$$\log_5(2x + 1) = 2.$$

$5^{\log_5(2x+1)} = 5^2$

$2x + 1 = 25$

$2x = 24$

$x = 12$

5. (16 points) Compute the derivative of the following functions.

(a) $f(x) = \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}$

$f'(x) = e^{e^{e^x}} e^{e^x} e^x$

(b) $f(x) = \cos(x \ln(x))$

$f'(x) = -\sin(x \ln x)\left(x\left(\frac{1}{x}\right) + \ln x\right)$

(c) $f(x) = \sqrt[3]{\sqrt{x^4}}$

$= \left[\left(x^4\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} = \left(x^2\right)^{\frac{1}{3}} = x^{\frac{2}{3}}$

$f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$

(d) $f(x) = \dfrac{\tan(x)}{x^2 + 1}$

$$f'(x) = \frac{(x^2+1)\sec^2 x - (\tan x)(2x)}{(x^2+1)^2}$$

6. (20 points) Evaluate the following limits. If a limit is infinite, say whether it is $+\infty$, $-\infty$, or neither.

(a) $\lim_{x\to 2^-} \dfrac{x^2+2x-8}{x^2-4x+4}$ $= \lim_{x\to 2^-} \dfrac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}(x-2)}$ gives $\frac{6}{0}$ by direct substitution, so this limit is infinite. Since $x \to 2^-$, $x<2$, so $x-2<0$. Since the numerator is positive and the denominator is negative, we get $-\infty$.

Caution! Try using L'Hopital's rule.

$= \lim_{x\to 2^-} \dfrac{2x+2}{2x-4} \neq \lim_{x\to 2^-} \dfrac{2}{2}$

not indeterminate

(b) $\lim_{x\to 0^+} x^x$

Set $y = x^x$

$\ln y = x\ln x$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$, which has the indeterminate form $\frac{-\infty}{\infty}$.

By L'Hopital's rule, $= \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$. Then $\lim_{x\to 0^+} e^{\ln y} = \lim_{x\to 0^+} y = e^0 = 1$.

(c) $\lim_{x\to\infty} \dfrac{e^x}{e^{2x}+1}$

$= \lim_{x\to\infty} \dfrac{\cancel{e^x}\ 1}{\cancel{e^x}\left(e^x + \frac{1}{e^x}\right)} = 0$

(d) $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$ $= \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}$

Since $x \to \infty$, $x>0$ and $\sqrt{x^2} = x$.

$= \lim_{x\to\infty} \dfrac{\cancel{x}\ 1}{\cancel{x}\sqrt{1+\frac{1}{x^2}}} = \dfrac{1}{\sqrt{1+0}} = 1$

(e) $\lim_{h\to 0} \dfrac{\sqrt{5+h}-\sqrt{5}}{h}$ $= f'(5)$ when $f(x) = \sqrt{x}$.

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f'(5) = \dfrac{1}{2\sqrt{5}}$

7. (6 points) (V1) Consider the function $f(x) = |x - 3| - 2$.

(a) Since $f(0) = 1$ and $f(4) = -1$, does the Intermediate Value Theorem guarantee that there is an x-value c in $(0, 2)$ so that $f(c) = 0$? (Only answer "Yes" or "No".) Yes

(b) Since $f(2) = -1$ and $f(4) = -1$, does the Mean Value Theorem guarantee that there is an x-value c in $(2, 4)$ so that $f'(c) = 0$? (Only answer "Yes" or "No".)

No. (a) is yes, because $f(0) > 0$ and $f(4) < 0$.

(V2) Consider the function $f(x) = |x + 1| - 2$.

(a) Since $f(-2) = -1$ and $f(2) = 1$, does the Intermediate Value Theorem guarantee that there is an x-value c in $(-2, 2)$ so that $f(c) = 0$? (Only answer "Yes" or "No".) Yes

(b) Since $f(-2) = -1$ and $f(0) = -1$, does the Mean Value Theorem guarantee that there is an x-value c in $(-2, 0)$ so that $f'(c) = 0$? (Only answer "Yes" or "No".) No

8. (10 points) Use logarithmic differentiation to find the ***second derivative*** of the following function:

$$f(x) = x^x$$

$y = x^x$

$\ln y = x \ln x$

$\frac{y'}{y} = \frac{x}{x} + \ln x$

$y' = x^x(1 + \ln x)$

$y'' = \frac{d}{dx}(x^x)(1+\ln x) + x^x\left(\frac{1}{x}\right)$

$= \underbrace{x^x(1+\ln x)}_{\text{from here}}(1+\ln x) + x^{x-1}$

9. (14 points) $f(x) = x^3 - 3x^2 - 9x + 4$

(a) Find all the critical numbers of f.

(b) Find the intervals over which f is increasing and over which f is decreasing.

(c) Find all local maximum and minimum values of f.

(d) Find all points of inflection of $y = f(x)$.

(e) Use the information in (a) through (d) to graph $y = f(x)$. You must plot the points at which local maximum and minimum values occur, any inflection points, and the y-intercept.

(a) $f'(x) = 3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$x = -1, 3$ are the critical #s.

(b) pos | neg | pos

-1 | 3

inc on $(-\infty, -1)$ and $(3, \infty)$

dec on $(-1, 3)$

(c) $f(-1) = -1 - 3 + 9 + 4 = 9$ is a local max value

$f(3) = -27 - 27 + 27 + 4 = -23$ is a local min value

(d) $f''(x) = 6x - 6 = 0$

$x - 1 = 0$

$x = 1$

neg | pos

1

$(1, -7)$ is an inflection pt.

(e) $f(0) = 4$