Math 141: Calculus I

Midterm 1 ANSWERS

October 23, 2014

1. (5 points) Find all values of x satisfying the inequality

$$|4x - 2| \geq 14.$$

- The given inequality holds when $4x - 2 \geq 14$ or when $4x - 2 \leq -14$.
- Solving the first inequality gives $x \geq 4$, and solving the second inequality gives $x \leq -3$.
- Therefore, the solution is $\boxed{x \leq -3 \text{ or } x \geq 4}$, which in interval notation is $\boxed{(-\infty, -3] \cup [4, \infty)}$.

2. (6 points) Find all values of t in the interval $[0, 2\pi]$ such that $\cos^2 t = \dfrac{1}{2}$.

- Taking the square root gives $\cos(t) = \pm\dfrac{1}{\sqrt{2}} = \pm\dfrac{\sqrt{2}}{2}$, which is one of the special angles.
- We see that there are four solutions in the interval $[0, 2\pi]$, namely $t = \boxed{\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}}$.

3. (12 points) Let $P = (3, 7)$ and $Q = (5, 3)$.

(a) Find the distance between P and Q.

- The distance formula gives $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(5-3)^2 + (3-7)^2} = \boxed{\sqrt{20} = 2\sqrt{5}}$.

(b) Find an equation for the line containing P and Q.

- The slope is $(y_2 - y_1)/(x_2 - x_1) = (3 - 7)/(5 - 3) = -2$.
- Since the line goes through $P = (3, 7)$, we can write the equation in point-slope form as $\boxed{y - 7 = -2(x - 3)}$.
- There are other ways of writing this answer, like $\boxed{y = -2x + 13}$ or $\boxed{2x + y = 13}$.

(c) Find an equation for the line through P perpendicular to the line $5x + 15y = 1$.

- We can rearrange the given line to the equation $y = (-1/3)x + 1/15$, so its slope is $-1/3$.
- Therefore, a line perpendicular to the given line must have slope $-1/(-1/3) = 3$.
- Using point-slope form, the line through P with slope 3 has equation $\boxed{y - 7 = 3(x - 3)}$.
- There are other ways of writing this answer, like $\boxed{y = 3x - 2}$ or $\boxed{-3x + y = -2}$.

4. (10 points) Find the following, in simplified form. (There should be no functions other than radicals and fractions in your answer.)

(a) The value of $\sin(\arctan(2/3))$.

- Let $\theta = \arctan(2/3)$. Then θ is the acute angle in a right triangle whose opposite side is 2 and whose adjacent side is 3. By the Pythagorean Theorem, the hypotenuse in this triangle is $\sqrt{2^2+3^2} = \sqrt{13}$.
- Then $\sin(\arctan(2/3)) = \sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \boxed{\dfrac{2}{\sqrt{13}}}$.

(b) The value of $\log_3(\sqrt{27})$.

- Since $\sqrt{27} = (3^3)^{1/2} = 3^{3/2}$, the logarithm to the base 3 is $\boxed{3/2}$.

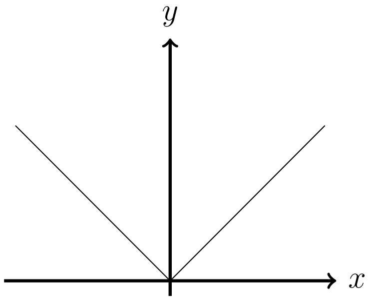
5. (6 points) Find all real numbers x such that $e^{2x} - 3e^x + 2 = 0$.

- Since $e^{2x} = (e^x)^2$, we can factor as $(e^x - 2)(e^x - 1) = 0$.
- This holds precisely when: $e^x = 2$ or $e^x = 1$.
- Taking the natural log of both sides yields the two solutions $x = \boxed{0, \ln(2)}$.

6. (6 points) Find all real numbers y such that $\log_2(y) + \log_2(2y) = \log_2(3y)$.

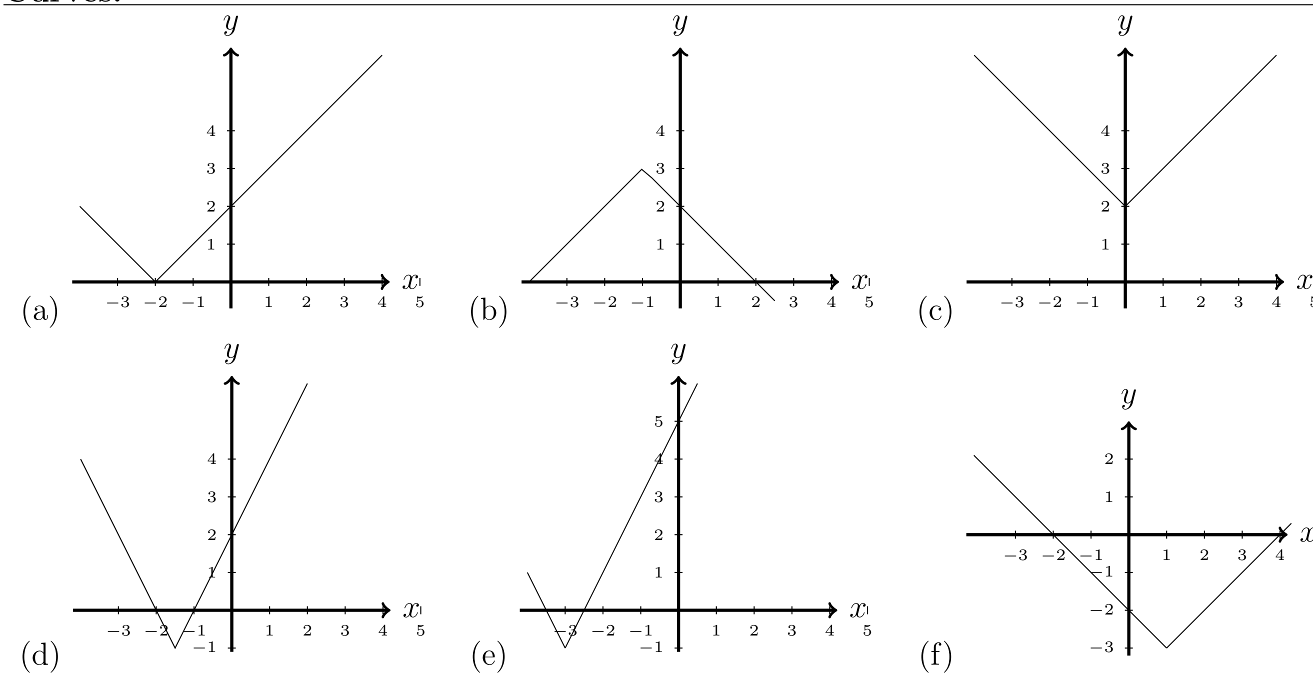
- Combine the logarithms to get $\log_2(2y^2) = \log_2(3y)$.
- Now exponentiate both sides to get $2^{\log_2(2y^2)} = 2^{\log_2(3y)}$, which simplifies to $2y^2 = 3y$.
- This equation has the two solutions $y = 0$ and $y = 3/2$.
- However, $y = 0$ does not satisfy the original equation, because $\log_2(0)$ is undefined. We can verify that $y = \boxed{3/2}$ does work, however, so it is the only solution.

7. (12 points) In the graph below, $y = |x|$.



Match the following curves to their equations.

Curves:



Equations:

1.) $y = \|x - 2\|$	4.) $y = -\|x\| + 2$	7.) $y = 3 + \|x + 1\|$
2.) $y = \|x + 2\|$	5.) $y = 3 - \|x + 1\|$	8.) $y = \|2x + 3\| - 1$
3.) $y = \|x\| + 2$	6.) $y = -3 + \|x - 1\|$	9.) $y = 2\|x + 3\| - 1$

Answers:

(a) 2 (b) 5 (c) 3 (d) 8 (e) 9 (f) 6

8. (12 points) Let $f(x) = x^2 - 4$ and $g(x) = \dfrac{2x}{x-1}$.

(a) Find $(f \circ g)(2) - (g \circ f)(2)$.

- We have $f(g(2)) = f(4) = 4^2 - 4 = 12$ and $g(f(2)) = g(0) = 0$, so the answer is $\boxed{12}$.

(b) Find the domain of $(f \circ g)(x)$.

- We have $(f \circ g)(x) = \left(\dfrac{2x}{x-1}\right)^2 - 4$.
- The only value of x for which this is undefined is $x = 1$, so the domain is $\boxed{x \neq 1}$, or, in interval notation, $\boxed{(-\infty, 1) \cup (1, \infty)}$.

Continue to set $f(x) = x^2 - 4$ and $g(x) = \dfrac{2x}{x-1}$.

(c) Find $g^{-1}(1)$.

- This is the value of x for which $g(x) = 1$, giving the equation $\dfrac{2x}{x-1} = 1$.
- Clearing the denominator gives $2x = x - 1$, so $\boxed{x = -1}$.

(d) Find all horizontal and vertical asymptotes of the graph of $y = g(x)$.

- For vertical asymptotes, we need to find the values of x where the function goes to $\pm\infty$. Since $g(x)$ is a quotient, this can only occur when the denominator is zero – namely, when $x = 1$. We can see that $\lim_{x \to 1^+} g(x) = \infty$, so $\boxed{x = 1}$ is the only vertical asymptote.
- For horizontal asymptotes, we compute

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x \cdot 2}{x \cdot (1 - 1/x)} = \lim_{x\to\infty} \frac{2}{1 - 1/x} = 2$$

and

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{x \cdot 2}{x \cdot (1 - 1/x)} = \lim_{x\to-\infty} \frac{2}{1 - 1/x} = 2$$

so $\boxed{y = 2}$ is a horizontal asymptote as $x \to \pm\infty$.

- Note: Asymptotes are lines, and it was necessary to give the actual equation of the line for full credit. Saying merely "1" or "2" is not correct.

9. (25 points) Compute the following limits. Note that ∞ and $-\infty$ are possible answers. **If a limit does not exist, you must explain why**.

(a) $\lim_{x\to 3} \frac{x-3}{x^2-4x+3}$

- We have $\lim_{x\to 3} \frac{x-3}{x^2-4x+3} = \lim_{x\to 3} \frac{x-3}{(x-3)(x-1)} = \lim_{x\to 3} \frac{1}{x-1} = \boxed{\frac{1}{2}}$.

(b) $\lim_{x\to 2^-} \frac{e^x}{\tan(\pi x)+\pi x}$

- The numerator and denominator are both continuous and defined at $x=2$, so the limit is just $\boxed{\frac{e^2}{\tan(2\pi)+2\pi} = \frac{e^2}{2\pi}}$.
- Note: For full credit, it was necessary to say that the function was continuous. (Essentially nobody did this.)

(c) $\lim_{w\to 3^-} \frac{7w-5}{w^2-9}$

- The denominator goes to zero, so we isolate the term that is approaching zero: $\lim_{w\to 3^-} \frac{7w-5}{(w-3)(w+3)} = \lim_{w\to 3^-} \frac{1}{w-3} \cdot \frac{7w-5}{w+3}$.
- As $w \to 3^-$, the first term goes to $-\infty$ while the second term goes to $\frac{7\cdot 3-5}{3+3} = \frac{16}{6}$, so the limit is $\boxed{-\infty}$.

(d) $\lim_{y\to 4} \dfrac{y-4}{\sqrt{2y+1}-\sqrt{y+5}}$

•We have

$$\begin{aligned}\lim_{y\to 4} \frac{y-4}{\sqrt{2y+1}-\sqrt{y+5}} &= \lim_{y\to 4} \frac{y-4}{\sqrt{2y+1}-\sqrt{y+5}} \cdot \frac{\sqrt{2y+1}+\sqrt{y+5}}{\sqrt{2y+1}+\sqrt{y+5}} \\ &= \lim_{y\to 4} \frac{(y-4)(\sqrt{2y+1}+\sqrt{y+5})}{y-4} \\ &= \lim_{y\to 4} \left(\sqrt{2y+1}+\sqrt{y+5}\right) = \boxed{\sqrt{9}+\sqrt{9}=6}.\end{aligned}$$

(e) $\lim_{x\to\infty} \dfrac{\sqrt{x^3+3x+5}}{x^2-4x-4}$

•We pull out the largest power of x from numerator and denominator separately:

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^3+3x+5}}{x^2-4x-4} &= \lim_{x\to\infty} \frac{\sqrt{x^3(1+3/x^2+5/x^3)}}{x^2(1-4/x-4/x^2)} \\ &= \lim_{x\to\infty} \frac{x^{3/2}}{x^2} \cdot \frac{\sqrt{1+3/x^2+5/x^3}}{1-4/x-4/x^2} \\ &= \lim_{x\to\infty} x^{-1/2} \cdot \frac{\sqrt{1+3/x^2+5/x^3}}{1-4/x-4/x^2}\end{aligned}$$

•Now as $x \to \infty$, the first term goes to 0 while the second term goes to $\dfrac{\sqrt{1}}{1} = 1$. Hence the limit is $\boxed{0}$.

•Note: Simply "zeroing out" everything but the largest terms is mathematically incorrect, and did not receive full credit.

10. (6 points) Find all real numbers c such that the function $f(x) = \begin{cases} cx^2 & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$ is continuous for all real numbers x.

- The two parts of the definition are each continuous for every real number x, so the only place there can be a problem is where the definition changes: namely, at $x = 2$.
- We need to compute the left limit, right limit, and value:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (cx^2) = 4c$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^3 - cx) = 8 - 2c$$

$$f(2) = 8 - 2c$$

- These will all be equal when $4c = 8 - 2c$, meaning $c = \boxed{\frac{8}{6} = \frac{4}{3}}$.
- Note: For full credit, it was necessary to write down the limits explicitly. Simply setting the two values equal and getting $c = 4/3$ was not enough.