Math 141: Calculus I

Midterm 2

November 18, 2014

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Indicate your instructor with a check in the appropriate box:

Prof. Kalyani Madhu	MWF 09:00 – 09:50 AM	
Prof. Alex Rice	TR 2:00 – 3:15 PM	
Prof. Saul Lubkin	MW 2:00 – 3:15 PM	
Prof. Evan Dummit	TR 4:50 – 6:05 PM	

- **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
- **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **You are responsible for checking that this exam has all 8 pages.**

QUESTION	VALUE	SCORE
1	20	
2	10	
3	12	
4	18	
5	24	
6	16	
TOTAL	100	

1. (20 points)

(a) Complete the following definition: For a function $f(x)$, we define a new function, called the *derivative* of $f(x)$, by the formula

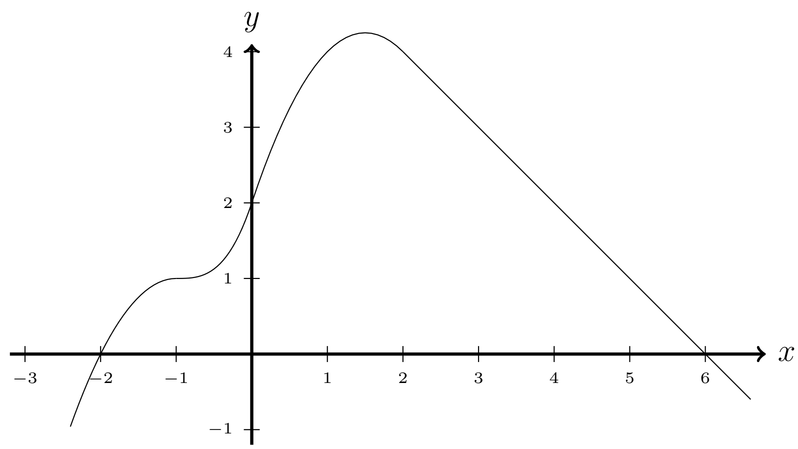
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

wherever this limit exists.

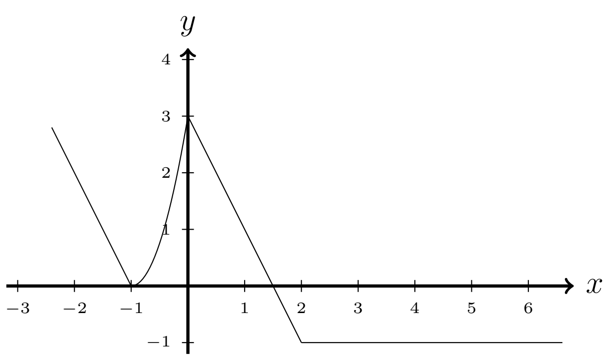
(b) USE THE DEFINITION ABOVE to find $f'(x)$ if $f(x) = \sqrt{1-2x}$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} \cdot \frac{\sqrt{1-2(x+h)} + \sqrt{1-2x}}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} \\
&= \lim_{h \to 0} \frac{(1-2(x+h)) - (1-2x)}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} \\
&= \lim_{h \to 0} \frac{1-2x-2h-1+2x}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} \\
&= \lim_{h \to 0} \frac{-2h}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} \\
&= \lim_{h \to 0} \frac{-2}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} \\
&= \frac{-2}{\sqrt{1-2x} + \sqrt{1-2x}} \\
&= \boxed{\frac{-1}{\sqrt{1-2x}}}.
\end{aligned}$$

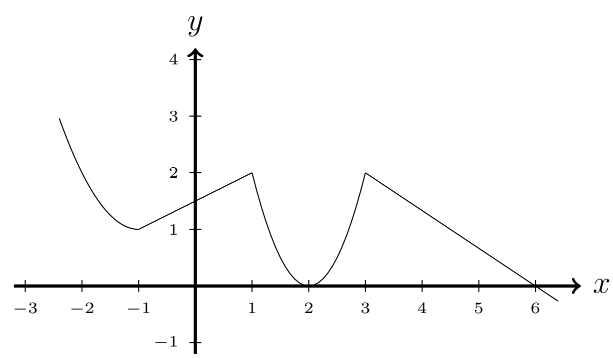
2. (10 points) In the graph below, $y = f(x)$, where $f(x)$ is an unknown piece-wise function.



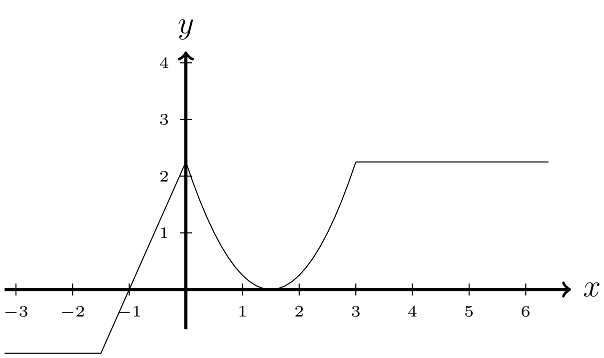
Which of these is the graph of $f'(x)$?



(a)



(b)



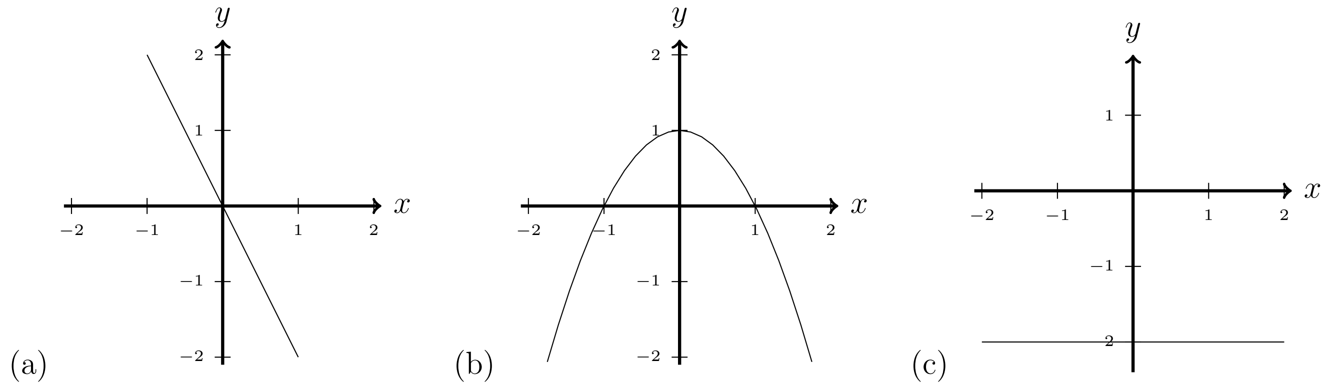
(c)

Answer: a

Explanation (not required, just here for reference): Notice that $f(x)$ has positively sloped tangent lines for x's in $(-\infty, -1)$, a horizontal tangent line at $x = -1$, then positively sloped tangent lines again between $x = -1$ and (roughly) $x = 1.5$, a horizontal tangent line at $x = 1.5$, and then negatively sloped tangent lines after that. Therefore, the y-coordinate on the graph of $f'(x)$ must start out positive, touch 0 at $x = -1$, go back to positive until crossing 0 at $x = 1.5$, then remain negative.

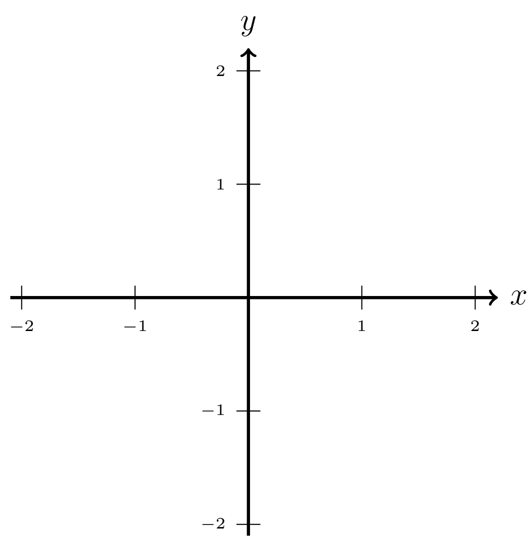
3. (12 points)

(a) The graphs below show $y = g(x), y = g'(x)$, and $y = g''(x)$ for some function $g(x)$. Determine which is which. (You don't need to try to figure out a formula for $g(x)$)



$g(x):$ b $\quad$ $g'(x):$ a $\quad$ $g''(x):$ c

(b) On the axes below, graph $y = g^{(3)}(x)$, the third derivative of g.



Note that from part (a) we know that $g''(x)$ is constant, so therefore $g^{(3)}(x)$ is just ZERO, and the graph is just the x-axis.

4. (18 points) Suppose that f and g are differentiable functions satisfying

$$f(2) = 0, \quad g(2) = a, \quad f'(2) = 1, \quad g'(2) = \pi, \quad f'(a) = 7$$

and assume $a \neq 0$.

(a) Let $h(x) = \frac{f}{g}(x)$. Find $h'(2)$ in terms of a.

By the quotient rule, $h'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \dfrac{a(1) - 0(\pi)}{a^2} = \boxed{\dfrac{1}{a}}$.

(b) Let $h(x) = (f \circ g)(x)$. Find $h'(2)$ in terms of a.

By the chain rule, $h'(2) = f'(g(2)) \cdot g'(2) = f'(a)g'(2) = \boxed{7\pi}$.

(c) Let $h(x) = g(x)e^{f(x)}$. Find $h'(2)$ in terms of a.

By the product rule and the chain rule,

$$h'(2) = g'(2)e^{f(2)} + g(2)e^{f(2)}f'(2) = \pi e^0 + ae^0(1) = \boxed{\pi + a}.$$

5. (24 points)

(a) Find the derivative of $f(x) = x^3 + \cos(x) - e^x$ with respect to x.

- By basic derivatives this is $\boxed{3x^2 - \sin(x) - e^x}$.

(b) Find $h'(x)$ if $h(x) = \sin(\sqrt{\tan(x)})$.

- By the Chain Rule we have

$$\begin{aligned} h(x) &= \cos(\sqrt{\tan(x)}) \cdot \frac{d}{dx}\left[\sqrt{\tan(x)}\right] \\ &= \boxed{\cos(\sqrt{\tan(x)}) \cdot \frac{1}{2}(\tan(x))^{-1/2}\sec^2(x)}. \end{aligned}$$

(c) Compute the value of $g''(1)$, where $g(t) = \tan^{-1}(3t)$.

- First we compute $g'(t) = \dfrac{1}{(3t)^2+1} \cdot 3 = \dfrac{3}{9t^2+1}$ by the Chain Rule.
- Then $g''(t) = \dfrac{0 \cdot (9t^2+1) - 3 \cdot 18t}{(9t^2+1)^2} = -\dfrac{54t}{(9t^2+1)^2}$.
- Evaluating, we have $g''(1) = \boxed{-\dfrac{54}{100}}$.

(d) Use logarithmic differentiation to find $\dfrac{dq}{dt}$ if $q = (t^5+1)^t$.

- We use logarithmic differentiation: $\ln(q(t)) = \sin(t) \cdot \ln(1+\sin(t))$.
- Differentiating both sides (using the product rule on the right) gives

$$\frac{q'(t)}{q(t)} = \cos(t) \cdot \ln(1+\sin(t)) + \sin(t) \cdot \frac{\cos(t)}{1+\sin(t)}.$$

- Thus, we get $q'(t) = \boxed{(1+\sin(t))^{\sin(t)} \cdot \left[\cos(t) \cdot \ln(1+\sin(t)) + \sin(t) \cdot \dfrac{\cos(t)}{1+\sin(t)}\right]}$.

6. (16 points) Consider the implicit curve $C : 3x^3 + 4xy + y^5 = 8$. (You may assume this defines y implicitly as a function of x.)

(a) Find the implicit derivative $\dfrac{dy}{dx}$.

- We differentiate both sides, remembering to view $y = y(x)$ as an implicit function of x, using the Product and Chain Rules.
- We get $9x^2 + 4y + 4x \cdot y' + 5y^4 y' = 0$.
- Rearranging gives $(4x + 5y^4) \cdot y' = -(9x^2 + 4y)$, so $y' = \boxed{-\dfrac{9x^2 + 4y}{4x + 5y^4}}$.

(b) Find an equation for the tangent line to the curve C at the point $(x, y) = (1, 1)$.

- From part (a), we get $y'(1, 1) = -\dfrac{9 + 4}{4 + 5} = -\dfrac{13}{9}$.
- Thus by point-slope, the tangent line has equation $\boxed{y - 1 = -\dfrac{13}{9}(x - 1)}$.